MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — March 2, 2018
Prof. Alan Guth

# PRELIMINARY PROBLEM SET 4

**DUE DATE:** Thursday, March 8, 2018, at 5:00 pm.

**TOPICS COVERED AND RELEVANT LECTURES:** This problem set covers aspects of the Schwarzschild metric, primarily following material presented in lecture.

**PRELIMINARY?!:** The full problem set, to be posted shortly, will include one more problem, involving the derivation of the geodesic equations from the equivalence principle.

**MAXIMUM GRADE:** This problem set has a total of 40 points so far, but the full problem set will have a little more.

## PROBLEM 1: COUNTING TENSORS *(5 pts)*

In lecture we derived the fact that if $S_{\mu_1 \cdots \mu_R}$ is fully symmetric in its indices, and each index can take on $D$ different values, then the number of independent components is

$$N = \frac{(R+D-1)!}{R!(D-1)!}\,.$$

Suppose that $A_{\mu_1 \cdots \mu_R}$ is fully antisymmetric in its indices, with each index again taking on $D$ different values. How many independent components does $A_{\mu_1 \cdots \mu_R}$ have?

## PROBLEM 2: CLOCKS ON EARTH AND IN SPACE *(10 pts)*

In this problem we consider the effects of relativity on a clock on the surface of a planet, using an idealized version of the Earth as an interesting example. Assume that the planet is perfectly spherical, and floats in empty space in a Schwarzchild geometry created by its own mass distribution. The planet has mass $M$, and the radius as measured by the Schwarzschild coordinates is $R$. The planet rotates in time $T$ as measured by an observer at infinity who is at rest relative to the Schwarzschild coordinates. (Here we will ignore the effect of the rotation on the metric, which we will learn about later in the course.) Now consider a clock (C) that lies on the surface of the planet at a point on the equator.

(a) [2 pts] Compute the time $T_{\mathrm{C,SR}}$ measured by the clock C after a single rotation, incorporating only the effects of special relativity. (That is, assuming that the clock is moving along a circular path of radius $R$ in Minkowski space and completes a circuit around the circle in time $T$ as measured by a stationary observer in Minkowski space.) To lowest order in $R/T$, by how much does this time differ from $T$ (i.e., find the time difference $\Delta T_{\mathrm{C,SR}} \equiv T_{\mathrm{C,SR}} - T$)? Using $R = R_\oplus = 6378.1$ km for the radius of the Earth and $T = 23$ hours, 56 minutes, and 4.09 seconds for the length of a sidereal day, what is the numerical value of this time difference?

(b) [2 pts] Compute the time $T_{\mathrm{C,GTD}}$ measured by the clock C after a single rotation of the planet, incorporating only the effect of gravitational time dilation from the metric component $g_{tt}$ in the Schwarzschild metric. (That is, calculate the time measured by a clock that travels relative to the surface of the planet, remaining stationary in the Schwarzschild coordinates.) To lowest order in $GM/R$, by how much does this time differ from $T$ (i.e., find the time difference $\Delta T_{\mathrm{GTD}} \equiv T_{\mathrm{C,GTD}} - T$)? Using $M_\oplus = 5.9724 \times 10^{24}$ kg as the mass of the Earth, and $G = 6.6741 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$, and $c = 2.998 \times 10^8$ m/s, what is the numerical value of the time difference?

(c) [2 pts] Compute the proper time $T_{\mathrm{Sch}}$ measured by the clock C along the prescribed trajectory in the full Schwarzschild metric. To lowest order in the small quantities indicated in parts (a) and (b), by how much does your result differ from the sum of the results from (a) and (b)? That is, compute

$$\Delta T_{(\mathrm{c})} \equiv (T_{\mathrm{Sch}} - T) - \Delta T_{(\mathrm{a})} - \Delta T_{(\mathrm{b})}$$

to lowest nonvanishing order in the small quantities $R/T$ and $GM/R$. Here $\Delta T_{(\mathrm{a})}$ and $\Delta T_{(\mathrm{b})}$ refer to the (correct) answers to parts (a) and (b), which means that they are each expanded only through the lowest order in their respective small quantities. Numerically, for the case of the Earth, how big is this difference.

(d) [2 pts] Compute the difference in time measured by a clock on Earth at sea level on the equator and a second clock on a mountain at 2000 m, also on the equator, over a period of one rotation of the planet. I.e., compute

$$\Delta T_{(d)} = T_{\mathrm{C,mountain}} - T_{\mathrm{C,sea\ level}}$$

to lowest nonvanishing order in small quantities, and evaluate it numerically for the Earth.

(e) [2 pts] Consider a geosynchronous satellite in a circular orbit above the equator, with an altitude of 42,164 km. Compute the time $T_{\mathrm{C,sat}}$ measured by a clock on the satellite during one rotation of the planet. Compute the time differences

$$\Delta T_1 \equiv T_{\mathrm{C,sat}} - T$$

and

$$\Delta T_2 \equiv T_{\mathrm{C,sat}} - T_{\mathrm{C,sea\ level}}\ .$$

In each case, give an expression to lowest nonvanishing order in the small quantities, and evaluate numerically for the Earth.

**PROBLEM 3: GRAVITATIONAL RED SHIFT** *(10 pts)*

In this problem we consider the gravitational redshift from two perspectives.

(a) [4 pts] The gravitational redshift can be seen as a direct consequence of Einstein's equivalence principle. Consider a photon that travels upward by a distance $d$ in Earth's near-surface gravitational field. Compute its redshift $\Delta\lambda/\lambda$ by computing the time $\Delta t$ needed for light to travel the distance $d$, the change in velocity $\Delta v$ that an inertial observer would experience compared to a stationary observer in the time $\Delta t$, and the Doppler shift when you boost into the stationary observer's frame. Express your answer in terms of $g, d$, and $c$.

(b) [6 pts] Now consider the same situation in general relativity. Compute the redshift by determining the relationship between the frequencies of emitted and received light at two heights differing by $d$. Show that in the Newtonian limit of Earth's near-surface field you reproduce the results of part (a).

**PROBLEM 4: RADIAL SCHWARZSCHILD GEODESICS** *(15 pts)*

Consider a particle falling radially inward (*i.e.,* at fixed $\theta, \phi$) along a geodesic in a Schwarzschild geometry with Schwarzschild radius $R_* = 2GM$. The particle has total energy $E = 1$, where

$$E = \left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \ .$$

Thus, the particle has the same energy as it would have if it were at rest at infinity. As $t \to -\infty$, such a geodesic has $r \to \infty$ and $dt/d\tau \to 1$.

(a) [8 pts] Use conservation of energy to compute $dt/d\tau$ and $dr/d\tau$ as functions of $r(\tau)$.

(b) [7 pts] Integrate to solve for $r(\tau)$. Compute the proper time difference experienced by the particle between the moments when it is at radius $r = 2R_*$ and at radius $r = R_*$. In class we learned that an observer at fixed radius $r$ will see an infalling object approach the horizon but never reach it. Does this calculation indicate that an infalling observer will have a different experience?