MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — April 6, 2018
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PROBLEM SET 8

DUE DATE: Thursday, April 12, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem concerns Raychaudhuri's equation, and is based on the class lectures and Carroll's Appendix F.

PROBLEM 1: THE RAYCHAUDHURI EQUATION, SPACELIKE AND TIMELIKE *(52 pts)*

Raychaudhuri's equation concerns a congruence of geodesics, where a congruence is a set of curves in an open region of spacetime such that every point in the region lies on precisely one curve. Let $T^\mu = \partial x^\mu/\partial\tau$ be the vector field of tangents to the geodesics $x^\mu(s_i,\tau)$, for a four-dimensional timelike or lightlike geodesic congruence. (Unlike Carroll's treatment, here we will do the two cases together.) Thus,

$$T_\mu T^\mu = \begin{cases} -1 & \text{(timelike)} \\ 0 & \text{(lightlike)} \end{cases} \tag{1}$$

and in both cases $T^\mu \nabla_\mu T^\nu = 0$. Let S_i^μ be a set of three vector fields denoting infinitesimal deviations between one geodesic and a nearby geodesic, defined by

$$S_i^\mu = \frac{\partial x^\mu}{\partial s_i} \ . \tag{2}$$

(a) *(5 pts)* Show that

$$\frac{\mathrm{D}S_i^\mu}{\mathrm{d}\tau} \equiv T^\nu \nabla_\nu S_i^\mu = B^\mu{}_\nu S_i^\nu \ , \tag{3}$$

where

$$B^\mu{}_\nu = \nabla_\nu T^\mu \ . \tag{4}$$

(b) *(3 pts)* Using only the geodesic equation and our original definition of the Riemann curvature tensor,

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \qquad \text{for any vector } V^\rho, \tag{5}$$

derive an expression for

$$\frac{\mathrm{D}B_{\mu\nu}}{\mathrm{d}\tau} \equiv T^\sigma \nabla_\sigma B_{\mu\nu} \tag{6}$$

in terms of $B_{\mu\nu}$, T^μ, and the curvature tensor. This is Carroll's Eq. (F.10), and was derived in class, but you are asked to go through the steps and verify that it works regardless of whether T^μ is timelike, lightlike, or spacelike.

(Note, however, that Carroll's Eq. (F.10) has some sign errors, and should be

$$\begin{aligned}\frac{\mathrm{D}B_{\mu\nu}}{\mathrm{d}\tau} \equiv U^\sigma \nabla_\sigma B_{\mu\nu} &= U^\sigma \nabla_\sigma \nabla_\nu U_\mu \\ &= U^\sigma \nabla_\nu \nabla_\sigma U_\mu + U^\sigma R^\lambda{}_{\mu\nu\sigma} U_\lambda \\ &= \nabla_\nu (U^\sigma \nabla_\sigma U_\mu) - (\nabla_\nu U^\sigma)(\nabla_\sigma U_\mu) + R_{\lambda\mu\nu\sigma} U^\sigma U^\lambda \\ &= -B^\sigma{}_\nu B_{\mu\sigma} + R_{\lambda\mu\nu\sigma} U^\sigma U^\lambda \ .\end{aligned} \tag{7}$$

Carroll's Eq. (F.11), however, the Raychaudhuri equation, is printed correctly:

$$\boxed{\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu \ .} \ ) \tag{8}$$

(c) *(4 pts)* Now break up $B_{\mu\nu}$ into symmetric and antisymmetric parts:

$$B_{\mu\nu} = B^S_{\mu\nu} + B^A_{\mu\nu} \ , \tag{9}$$

where

$$B^S_{\mu\nu} \equiv B_{(\mu\nu)} \ , \qquad B^A_{\mu\nu} \equiv B_{[\mu\nu]} \equiv \omega_{\mu\nu} \ . \tag{10}$$

Use your result from part (b) to derive an equation for $\mathrm{D}B^A_{\mu\nu}/\mathrm{d}\tau$. It should have the property that if $B^A_{\mu\nu}$ vanishes at one point on the geodesic, then it vanishes everywhere on the geodesic.

Carroll points out that the vanishing of $B^A_{\mu\nu} \equiv \omega_{\mu\nu}$ will happen if and only if T^μ is hypersurface-orthogonal, but he does not seem to show that hypersurface-orthogonal families of geodesics can always be constructed. In part (c) you showed that if $B^A_{\mu\nu} = 0$ vanishes at one point on the geodesic, then it vanishes everywhere on the geodesic. Here we will use that fact to show that an irrotational ($B^A_{\mu\nu} = 0$) congruence of timelike geodesics can always be constructed.

Consider a (3+1)-dimensional spacetime with metric $g_{\mu\nu}(x)$ that contains a spacelike hypersurface Σ (i.e., a three-dimensional surface whose normal vector is everywhere timelike). The surface can be infinite or finite. Suppose that coordinates are chosen for the four-dimensional space so that

$$x^0 = 0 \qquad \text{on the surface } \Sigma. \tag{11}$$

We will construct an irrotational congruence of timelike geodesics by first specifying the congruence on Σ, with $B^A_{\mu\nu} = 0$ on Σ. The geodesics that intersect Σ can then be extended away from Σ, and the region they fill will then be guaranteed to contain an irrotational geodesic congruence.

To construct the vector field T^μ, we first choose a smooth scalar function $\phi(x^i)$ defined on Σ, and consider a covariant vector field T_μ for which the spacelike components ($i = 1, 2,$ or 3) are given on Σ by

$$T_i = \frac{\partial \phi}{\partial x^i} \ . \tag{12}$$

(d) *(3 pts)* Show that for spacelike indices i, j,

$$\nabla_i T_j = \nabla_j T_i \qquad (\text{on } \Sigma). \tag{13}$$

(e) *(5 pts)* The last component T_0 can be determined on Σ by normalization, $g^{\mu\nu} T_\mu T_\nu = -1$. This will give a quadratic equation for T_0. Show that this quadratic equation always has real solutions for T_0. (Hint: for any given point on Σ, imagine transforming to locally inertial coordinates centered on that point.)

(f) *(5 pts)* To complete the demonstration that $B^A_{\mu\nu} = 0$, use the facts that T is normalized and that it satisfies the geodesic equation $T^\mu \nabla_\mu T_\nu = 0$ to show that

$$\nabla_i T_0 = \nabla_0 T_i \qquad (\text{on } \Sigma). \tag{14}$$

(Hint: first show that normalization implies that T^0 cannot vanish. Therefore the truth or falsehood of Eq. (14) is not changed by multiplying it by T^0. Then see what you can learn by using normalization and the geodesic equation.)

(g) *(3 pts)* Take the trace of the equation for $\mathrm{D}B_{\mu\nu}/\mathrm{d}\tau$ that you derived in part (b), to obtain an equation for

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \ , \tag{15}$$

where $\theta \equiv B^\mu{}_\mu = \nabla_\mu T^\mu$. Use Eq. (9) to express the right-hand side of the equation in terms of $B^S_{\mu\nu}$, $B^A_{\mu\nu}$, T^μ, and the curvature tensor.

Comment: If one thinks of the geodesic congruence as describing a fluid flow, then θ describes the divergence ($\theta > 0$) or convergence ($\theta < 0$) of the flow. One might be puzzled why θ is a four-divergence, while the intuitive concept of a divergence would correspond to a three-divergence of the spatial vector T^i. But remember that $T^\mu \nabla_\mu T^\nu = 0$, so if we describe the system in the locally inertial frame in which the fluid is at rest, so $T^\mu = (1, 0, 0, 0)$, then $\partial_0 T^0 = 0$ and $\theta = \partial_i T^i$. So θ represents the divergence of the fluid in the locally inertial frame.

For the timelike case, the next step in the standard treatment is to introduce the projector onto the subspace orthogonal to T^μ,

$$P^\mu{}_\nu = \delta^\mu_\nu + T^\mu T_\nu \ , \tag{16}$$

and then decompose $B^S_{\mu\nu}$ as

$$B^S_{\mu\nu} \equiv \frac{1}{3} \theta P_{\mu\nu} + \sigma_{\mu\nu} \ , \tag{17}$$

where $\sigma_{\mu\nu}$, the shear, is symmetric, traceless, and orthogonal to T^μ (i.e., $T^\mu \sigma_{\mu\nu} = 0$). Then, if we assume that $B^A_{\mu\nu} = 0$, the quantity

$$B^2 \equiv B^S_{\mu\nu} B^{S,\mu\nu} \ , \tag{18}$$

which appears on the right-hand side of the equation for $\mathrm{d}\theta/\mathrm{d}\tau$, can be bounded by

$$B^2 = \frac{1}{3}\theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} \geq \frac{1}{3}\theta^2 \ . \tag{19}$$

This leads finally to the conclusion that timelike irrotational congruences obey the equation

$$\boxed{\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \leq -\frac{1}{3}\theta^2 - R_{\mu\nu}T^\mu T^\nu \ .} \tag{20}$$

For the lightlike case this does not work out as easily, because there is no good analogue to $P^\mu{}_\nu$. Carroll handles this by introducing a frame-dependent projection tensor $Q_{\mu\nu}$, and then at the end shows (or asks the reader to show) that his result for $\mathrm{d}\theta/\mathrm{d}\tau$ is nonetheless frame-independent. Here we will follow a different approach, using Lagrange multipliers to minimize $B^S_{\mu\nu}B^{S,\mu\nu}$ subject to the relevant constraints.

(h) *(3 pts)* Show that $T^\mu B^S_{\mu\nu} = 0$.

(i) *(8 pts)* Minimize $B^2 \equiv B^S_{\mu\nu}B^{S,\mu\nu}$ subject to the constraints

$$T^\mu B^S_{\mu\nu} = 0 \tag{21}$$

$$g^{\mu\nu} B^S_{\mu\nu} = \theta \ , \tag{22}$$

by using Lagrange multipliers. Consider the quantity

$$L = B^2 + \lambda^\nu T^\mu B^S_{\mu\nu} + \bar{\lambda}(g^{\mu\nu}B^S_{\mu\nu} - \theta) \ , \tag{23}$$

which produces the constraints (21) and (22) when varied with respect to λ^ν and $\bar{\lambda}$, respectively. For the timelike case, use this equation to find the minimum value of B^2 subject to the constraints. (Hint: vary L with respect to $B^S_{\mu\nu}$ and obtain an expression for $B^S_{\mathrm{min},\mu\nu}$ in terms of the Lagrange multipliers, where $B^S_{\mathrm{min},\mu\nu}$ is the value of $B^S_{\mu\nu}$ which minimizes B^2 subject to the constraints. Then impose the constraints to find the values of the Lagrange multipliers. Finally, evaluate B^2_{min}. To match the result in Eq. (19), you should find $B^2_{\mathrm{min}} = \frac{1}{3}\theta^2$. You may find it a little tricky to vary L with respect to $B^S_{\mu\nu}$, since there are various ways of imposing the fact that $B^S_{\mu\nu}$ is symmetric. One approach is to define $\delta L/\delta B^S_{\mu\nu}$ as the symmetric matrix that satisfies

$$\delta L = \frac{\delta L}{\delta B^S_{\mu\nu}} \delta B^S_{\mu\nu} \tag{24}$$

when $B^S_{\mu\nu}$ is varied by an infinitesimal symmetric matrix $\delta B^S_{\mu\nu}$.)

(j) *(5 pts)* Minimize B^2 subject to the constraints of Eqs. (21) and (22) for the lightlike case, $T^2 = 0$. You should find that in this case the values of the Lagrange multipliers are not completely determined, but the value of B^2_{min} is determined to be $\frac{1}{2}\theta^2$.

The term on the right-hand side of the Raychaudhuri equation that depends on the curvature has the form $-R_{\mu\nu}T^\mu T^\nu$. The Einstein equation relates $R_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{25}$$

where G is Newton's gravitational constant. By taking the trace of this equation and substituting back into the equation, it can be rewritten as

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda\right) \ . \tag{26}$$

Thus, for the timelike case, the term will always contribute negatively to the right-hand side if

$$\left(T_{\mu\nu} - \frac{1}{2}T^\lambda{}_\lambda g_{\mu\nu}\right)T^\mu T^\nu \geq 0 \tag{27}$$

for all timelike vectors T^μ, which is called the strong energy condition. For a perfect fluid, the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \ , \tag{28}$$

where ρ is the energy density, p is the pressure, and $U^\mu \equiv T^\mu$ is the four-velocity of the fluid.

(k) *(5 pts)* Show that the strong energy condition, for a perfect fluid, is equivalent to

$$\rho + p \geq 0 \qquad \text{and} \qquad \rho + 3p \geq 0 \ . \tag{29}$$

(l) *(3 pts)* For the lightlike case, the term $-R_{\mu\nu}T^\mu T^\nu$ on the right-hand side of the Raychaudhuri equation will always contribute negatively provided that

$$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \tag{30}$$

for all lightlike vectors ℓ^μ, which is called the null energy condition. Show that for a perfect fluid, the null energy condition is equivalent to

$$\rho + p \geq 0 \ . \tag{31}$$