**PROBLEM 1: THOUGHT EXPERIMENTS FOR THE RELATIVITY OF SINTANESCY**

(15 pts)

In the first lecture we discussed thought experiments that can be used to derive the three kinematic consequences of special relativity: time dilation, Lorentz contraction, and the relativity of simultaneity. In this problem we will carry out two versions of a thought experiment to derive the relativity of simultaneity. You may assume that we already know about time dilation, Lorentz contraction, and the fact that light always travels at speed $c$ in any inertial frame to derive the relativity of simultaneity:

- **TIME DILATION:** Any clock which is moving at speed $v$ relative to an inertial reference frame will "appear" (to an observer using that reference frame) to run slower than normal by a factor $\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$, where $\beta \equiv v/c$.

- **LORENTZ-FITZGERALD CONTRACTION:** Any rod which is moving at a speed $v$ along its length relative to an inertial reference frame will "appear" (to an observer using that reference frame) to be shorter than its normal length by the same factor $\gamma$. A rod which is moving perpendicular to its length does not undergo a change in apparent length.

To discuss the relativity of simultaneity, consider a high-speed train of rest length $\ell_0$ that travels on a straight track at speed $v$ relative to the ground (with $v$ comparable to $c$), with a clock at each end.

(a) [5 pts] Suppose the clocks are synchronized in the rest frame of the train by using a light pulse sent from the front of the train to the rear. The pulse leaves the clock at the front when it reads $t_{\text{clock 1}} = 0$. When it arrives at the clock in the rear, the clock is set to $t_{\text{clock 2}} = t_{\text{clock 1}} + \frac{\ell_0}{c}$. Using time dilation, Lorentz contraction, and the fact that light always travels at speed $c$, calculate the time difference between the two clocks as measured in the frame of the ground. (Remember that we are imagining that we have not yet derived the formula for the Lorentz transformation, so we cannot use it here.)

(b) [10 pts] Now consider the same train, but consider another way that the clocks can be synchronized. Suppose that both clocks start at the front of the train, and are both considered to be in the same frame. How can the clocks be synchronized so that they will remain synchronized in the frame of the ground?

**REVIEW:** Any clock which is moving at speed $v$ relative to an inertial reference frame will "appear" (to an observer using that reference frame) to run slower than normal by a factor $\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$.

**PROBLEM SET 1 SOLUTIONS**

Due Date: Thursday, February 15, 2018, at 5:00 pm. Put your solutions in the homework box in Bldg 8, 3rd floor, at the intersection with the 4th floors of Bldg 16 and Bldg 26.

Reading: Carroll, Chapter 1, Sections 1.1 - 1.8. Note: we will go into the mathematical formalism of special relativity in more detail in later weeks, for the present a loose understanding of these ideas will suffice.
We can now take the limit of \( \frac{\gamma^2 - 1}{\gamma} \) to get the numerator and denominator of this expression:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{(\gamma - 1)(\gamma + 1)}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(11)

where \( \gamma \) is defined in terms of the clock speed. The difference between the times on the two clocks is then:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(0)

When the rear clock reads the back of the train, the two clocks show in the frame of the ground their different times due to the proper length of the train in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(6)

The difference in time between the two clocks is due to the different times they show in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(8)

The difference in time between the two clocks is due to the different times they show in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(1)

When the rear clock reads the back of the train, the two clocks show in the frame of the ground their different times due to the proper length of the train in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(9)

When the rear clock reads the back of the train, the two clocks show in the frame of the ground their different times due to the proper length of the train in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(2)

When the rear clock reads the back of the train, the two clocks show in the frame of the ground their different times due to the proper length of the train in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(3)

When the rear clock reads the back of the train, the two clocks show in the frame of the ground their different times due to the proper length of the train in the frame of the ground. Because the train clock has a different proper time than the clock on the train, the apparent difference in time is:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

(4)

Thus, the trailing clock reads later than the leading clock by an amount:

\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]

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\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
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\[
\frac{\gamma^2 - 1}{\gamma} = \frac{\gamma - 1}{1} = \gamma - 1
\]
A = τ

\[ A = \frac{1}{\gamma} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]

\[ \frac{\tau}{2} = \frac{\gamma}{\gamma} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]

\[ \frac{\tau}{2} = \frac{1}{\gamma} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \]

...
system, by an amount $\gamma vL/c^2$. He would then travel a distance $\gamma L$, at speed $v$, which takes a time $\gamma L/v$. Thus, when observer $A$ returns, the clocks of observer $B$ will read

$$\tau_B(\text{as computed by } B) = \frac{\gamma L}{v} - \frac{\gamma vL}{c^2} = \frac{\gamma L}{v} \left(1 - \frac{v^2}{c^2}\right) = \frac{L}{\gamma v},$$

in agreement with Eq. (16). When $B$ tries to understand the time measured on $A$’s clock, he sees a simple picture in which $A$ travels a distance $\gamma L$ at speed $v$, while $A$’s clock runs slowly by a factor of $\gamma$, so the time interval is

$$\tau_A(\text{as computed by } B) = \frac{1}{\gamma} \frac{L}{v} = \frac{L}{c},$$

in agreement with Eq. (14). For further explanations, see:


PROBLEM 3: LORENTZ VECTORS (8 pts)

(a) [2 pts] Show that for two timelike separated events, there is an inertial frame in which $\Delta t \neq 0$, $\Delta \bar{x} = 0$.

(b) [2 pts] Show that for two spacelike separated events, there is an inertial frame in which $\Delta t = 0$, $\Delta \bar{x} \neq 0$.

(c) [2 pts] Show that the sum of any two orthogonal spacelike vectors is spacelike.

(d) [2 pts] Show that a timelike vector and a null vector cannot be orthogonal.

ANSWER:

Without loss of generality, let us consider a 2D spacetime for parts (a) and (b) in which frame $S'$ is traveling at speed $v$ with respect to an inertial frame $S$. Then the standard Lorentz transformation (with $c = 1$) is

$$\Delta t' = \gamma (\Delta t - vt),$$
$$\Delta \bar{x}' = \gamma (\Delta x - v\Delta t).$$

(a) For a timelike interval, we have $\Delta x^2 = -\Delta t^2 + \Delta \bar{x}^2 < 0$, so that $\Delta x^2 < \Delta t^2$. Looking at Eq. (19), we see that the condition $\Delta x' = 0$ is fulfilled by setting $v = \Delta x/\Delta t < 1$. Plugging this expression for $v$ into the equation for $\Delta t'$, we find

$$\Delta t' = \gamma \left(\Delta t - \frac{\Delta x}{\Delta x'} \Delta \bar{x}\right) = \frac{\gamma}{\Delta x} (\Delta t^2 - \Delta \bar{x}^2) \neq 0.$$

Hence, $\Delta t' \neq 0$ and $\Delta x' = 0$ for $v = \Delta x/\Delta t$.

By expanding

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - \Delta x^2/\Delta t^2}},$$

we can further simplify this to

$$\Delta t = \frac{1}{\Delta t} (\Delta t^2 - \Delta \bar{x}^2) = \sqrt{\Delta t^2 - \Delta \bar{x}^2},$$

which is the form guaranteed by the invariance of the spacetime interval.

For a spacelike interval, we have $\Delta x^2 = -\Delta t^2 + \Delta \bar{x}^2 > 0$, so that $\Delta x^2 > \Delta t^2$. Looking at Eq. (19), we see that the condition $\Delta x' = 0$ is fulfilled by setting $v = \Delta x/\Delta t > 1$. Plugging this expression for $v$ into the equation for $\Delta x'$, we find

$$\Delta x' = \gamma \left(\Delta x - \frac{\Delta t}{\Delta x} \Delta \bar{x}\right) = \frac{\gamma}{\Delta x} (\Delta x^2 - \Delta t^2) \neq 0.$$

Hence, $\Delta t' = 0$ and $\Delta x' \neq 0$ for $v = \Delta t/\Delta x$.

As in Part (a), we can simplify this to

$$\Delta x' = \frac{1}{\Delta x} (\Delta x^2 - \Delta t^2) = \sqrt{\Delta x^2 - \Delta t^2},$$

as guaranteed by the invariance of the spacetime interval.

For parts (c) and (d), consider two orthogonal vectors $x^a$ and $y^a$. Orthogonality requires that

$$0 = \eta_{ab}x^b y^a = -x^0y^0 + \mathbf{x} \cdot \mathbf{y}$$

Then the spacetime interval of the sum of the orthogonal vectors is

$$\Delta s_{x+y}^2 = \eta_{ab}(x^a + y^a)(x^b + y^b)$$
$$= \eta_{ab}x^ax^b + \eta_{ab}y^ay^b + 2\eta_{ab}x^ay^b$$
$$= \Delta s_x^2 + \Delta s_y^2.$$
\[
\left(\frac{x^0}{x^i} - 1\right) \cdot \beta^i = 0
\]

\[\text{ANSWER:} \]

\(9.63 \text{ General Relativity Problem set 1 Solutions, Spring 2018 p. 9}\)
so the times for the four events would be  
\[ t' = 0, \quad -\gamma v_x/c^2, \quad -\gamma v_y/c^2, \quad -\gamma v_z/c^2. \]
Since \( v_x, v_y, v_z \) can be chosen arbitrarily, limited only by \( \|v\| < c \), any ordering can be achieved.

**(PROBLEM 5: LORENTZ INDEX MANIPULATION)**

*(11 pts)*

Carroll Problem 1.7: Imagine we have a tensor \( X_{\mu\nu} \) and a vector \( V^\mu \), with components

\[
X_{\mu\nu} = \begin{pmatrix}
20 & 1 \\
-1 & 0 \\
-103 & 2 \\
-110 & 0 \\
-211 & -2
\end{pmatrix}
V^\mu = (-1, 2, 0, -2).
\]

Find the components of

(a) \( X_{\mu\nu}^\prime \)

(b) \( X_{\nu\mu} \)

(c) \( X^\mu_{\nu\nu} \)

(d) \( X^\mu_{\nu\nu} \)

(e) \( X_{\lambda\lambda} \)

(f) \( V^\mu X^\mu \)

(g) \( X^\mu_{\nu\nu} V^\mu \)

Note that \( X^\mu_{\nu\nu} \), \( X^\mu_{\nu\nu} \) denote tensors that are symmetrized and antisymmetrized on the indices.

\(\eta_{\mu\nu}\) is the Minkowski metric.

\[
X_{\mu\nu} = \frac{1}{2} (X_{\mu\nu} + X_{\nu\mu}) = \frac{1}{2} (X_{\mu\nu} + (X_{\mu\nu})^T)
\]

\[
X^\mu_{\nu\nu} = \frac{1}{2} (X_{\mu\nu} - X_{\nu\mu}) = \frac{1}{2} (X_{\mu\nu} - (X_{\mu\nu})^T)
\]

We will display our results in multiple ways, in order to demonstrate the various ways of looking at the operations being performed.

**(ANSWER):**

Throughout this problem, we will write the expressions in multiple ways, in order to demonstrate the various ways of looking at the operations being performed.

(a) Using the Minkowski metric \( \eta_{\mu\nu} \), we find

\[
X_{\mu\nu} = \begin{pmatrix}
20 & 1 \\
-1 & 0 \\
-103 & 2 \\
-110 & 0 \\
-211 & -2
\end{pmatrix}
\]

\[
X_{\mu\nu} = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

(b) Again making use of the Minkowski metric to lower an index, we find

\[
X_{\nu\mu} = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

\[
X_{\nu\mu} = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

(c) Symmetrizing, we find

\[
X^\mu_{\nu\nu} = \frac{1}{2} (X_{\mu\nu} + (X_{\mu\nu})^T) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 3 & 2 \\
0 & 0 & -2 & 0
\end{pmatrix}
\]

(d) Antisymmetrizing, we find

\[
X^\mu_{\nu\nu} = \frac{1}{2} (X_{\mu\nu} - (X_{\mu\nu})^T) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(e) Using the Minkowski metric

\[
X^\mu_{\nu\nu} = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

\[
X^\mu_{\nu\nu} = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

(f) \( V^\mu X^\mu = (-1, 2, 0, -2) \begin{pmatrix}
20 & 1 \\
-1 & 0 \\
-103 & 2 \\
-110 & 0 \\
-211 & -2
\end{pmatrix}
\]

\[
V^\mu X^\mu = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

\[
V^\mu X^\mu = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

(g) \( X^\mu_{\nu\nu} V^\mu = \begin{pmatrix}
2 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 3 & 2 \\
0 & 0 & -2 & 0
\end{pmatrix} \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

\[
X^\mu_{\nu\nu} V^\mu = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]

\[
X^\mu_{\nu\nu} V^\mu = \begin{pmatrix}
-1 & 1 \\
0 & 1 \\
0 & -1 \\
1 & 0 \\
-1 & 0
\end{pmatrix}
\]
\(\phi\) c = \(\phi\) sinh s \(\phi\) c = \(\phi\) (\(\phi\) + \(\phi\) - \(\phi\) - \(\phi\)) = \(\phi\) (\(\phi\) + \(\phi\) - \(\phi\) - \(\phi\))

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

We have five solutions to this equation, which is given by:

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

We can then transform:

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

We have the Lorentz transformation matrix for a boost of speed \(\vec{v}\) in the \(-\vec{z}\) direction, which is given by:

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

We must verify the Lorentz transformation matrices for a boost of speed \(\vec{v}\) in the \(-\vec{z}\) direction, which is given by:

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

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\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

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\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]

We have the Lorentz transformation matrix for a boost of speed \(\vec{v}\) in the \(-\vec{z}\) direction, which is given by:

\[\begin{pmatrix}
0 & \phi \\
\phi & 0
\end{pmatrix}
\]
with a pure electric field in the $S'$ frame. Thus, the Lorentz transformation for a boost in the $x$-direction is given by

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \gamma' \gamma \begin{pmatrix}
\cosh \theta & \sinh \theta & 0 & 0 \\
\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}^T$$

This transformation, known as the Lorentz transformation, is used to transform coordinates and velocities between different reference frames. The direction of motion is given by $\gamma' \gamma \theta$.

### Gauss' Law and Ampere's Law

Applying Gauss' law and Ampere's law, the fields measured in $S'$ are expected to be

$$\begin{pmatrix}
\Phi_a & \sigma_a \\
\sigma_a & \rho
\end{pmatrix} = \begin{pmatrix}
\frac{\sigma_a}{\rho} & \Phi \\
\Phi & \rho
\end{pmatrix} = \begin{pmatrix}
\mathbf{a} & \mathbf{A} \\
\mathbf{A} & \mathbf{B}
\end{pmatrix}$$

and the surface current density is

$$\mathbf{J} = \frac{\partial \mathbf{A}}{\partial t} = \mathbf{J}$$

### Example Calculation

Consider a charged particle moving in a magnetic field. The Lorentz force acting on the particle is given by

$$\mathbf{F} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B}$$

where $q$ is the charge of the particle, $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the velocity of the particle, and $\mathbf{B}$ is the magnetic field. This force is Lorentz-covariant, meaning it transforms according to

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \gamma' \gamma \begin{pmatrix}
\cosh \theta & \sinh \theta & 0 & 0 \\
\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}^T$$

This ensures that the laws of electromagnetism remain valid in all inertial frames.