Problem Set 8 Solutions

Physics 8.962: General Relativity

Massachusetts Institute of Technology

Thursday, April 12, 2018, at 5:00 pm.

DUE DATE: Thursday, April 12, 2018, at 2:00 pm

Answer:

\[ (\nabla \sigma \Delta + \frac{\partial \sigma}{\partial \tau}) \Delta = \nabla \sigma \Delta \]

where

\[ \Delta ^{\sigma} = \frac{\partial}{\partial \tau} \]

Show that

\[ \frac{\partial S}{\partial \sigma} = \nabla \Delta \]

The Raychaudhuri equation, spacetime and time-like.

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Consider a $(3+1)$-dimensional spacetime with metric $g_{\mu\nu}$, and let $A$ be a 1-form on the spacetime, expressed as $A = A^\mu dx^\mu$. The covariant derivative of a tensor field $T^\mu\nu$ is defined by $\nabla_\lambda T^\mu\nu = \frac{\partial T^\mu\nu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\sigma}T^\sigma\nu$, where $\Gamma^\mu_{\lambda\sigma}$ are the Christoffel symbols.

The Ricci tensor $R_{\mu\nu}$ is defined as $R_{\mu\nu} = \nabla_\lambda g^\lambda\mu g^\tau\nu \nabla_\tau g_{\lambda\tau} - \nabla_\lambda g^\lambda\mu g^\tau\nu \nabla_\tau g_{\lambda\tau}$. The Einstein tensor $G_{\mu\nu}$ is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, where $R$ is the scalar curvature.

The Ricci scalar $R$ is defined as the trace of the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$. The Einstein equations in Einstein-Cartan theory are given by $G_{\mu\nu} = 
abla_\lambda g^\lambda\mu g^\tau\nu \nabla_\tau g_{\lambda\tau} - \nabla_\lambda g^\lambda\mu g^\tau\nu \nabla_\tau g_{\lambda\tau} = 8\pi T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor.

Note that in General Relativity, the Christoffel symbols are related to the metric tensor by $\Gamma^\mu_{\lambda\sigma} = \frac{1}{2}g^{\mu\rho}(\frac{\partial g_{\rho\lambda}}{\partial x^\sigma} + \frac{\partial g_{\rho\sigma}}{\partial x^\lambda} - \frac{\partial g_{\lambda\sigma}}{\partial x^\rho})$.
\begin{align*}
\frac{\partial p}{\partial \theta} &= \sigma^B \sigma^B \frac{\partial p}{\partial \theta} - \frac{\partial p}{\partial \theta} = \sigma^B \sigma^B g_{\theta B} = \sigma^B \sigma^B g_{\theta B} \\
\text{Using the Leibnitz rule, so we have the derivative on the left side, the second \( \sigma^\theta \) and the term on the right.}
\end{align*}

The derivative is given by:

\[ \frac{\partial p}{\partial \theta} = \sigma^B \sigma^B \frac{\partial p}{\partial \theta} - \frac{\partial p}{\partial \theta} = \sigma^B \sigma^B g_{\theta B} = \sigma^B \sigma^B g_{\theta B} \]

We can continue the discussion by multiplying by \( \sigma^\theta \) and go back to the previous step.
One approach is to define $\theta$ symmetric, since there are various ways of imposing the fact that $S$ is symmetric. But remember that $\theta = 0$ (21), so if we define the quantity

$$\theta = \frac{\partial}{\partial \theta} B = 0,$$

where the function $\theta$ is the shear, where $\theta$ is a function of $B$, subject to the constraint $\theta = 0$. Then, the quantity $\theta$ is the shear, and we have the expression

$$\theta = \frac{\partial}{\partial \theta} B + \frac{\partial}{\partial \theta} L = 0.$$

We have the constraints subject to the constraint $\theta = 0$. The quantity $\theta$ is the shear, where $\theta$ is a function of $B$, subject to the constraint $\theta = 0$. Then, the quantity $\theta$ is the shear, and we have the expression

$$\theta = \frac{\partial}{\partial \theta} B + \frac{\partial}{\partial \theta} L = 0.$$

We have the constraints subject to the constraint $\theta = 0$. The quantity $\theta$ is the shear, where $\theta$ is a function of $B$, subject to the constraint $\theta = 0$. Then, the quantity $\theta$ is the shear, and we have the expression

$$\theta = \frac{\partial}{\partial \theta} B + \frac{\partial}{\partial \theta} L = 0.$$
are not completely determined, but the value of \(B\) is determined to be 0. You should find that in this case the values of the Lagrange multipliers that satisfy the constraints of Eqs. (21) and (22) for the Einstein equations subject to the conditions of Eqs. (9) are

\[
\left(\varepsilon L \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \epsilon} \right) \delta s = 0
\]

where \(\varepsilon\) is the Newton's gravitational constant. By taking the trace of this equation and sub...

\[
' \theta = \varepsilon L \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \epsilon} = \varepsilon L \cdot \chi - \chi
\]

where \(\chi\) and the vector \(\epsilon\) have not been determined. But then

\[
' \theta = \varepsilon L \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \epsilon} = \varepsilon L \cdot \chi - \chi
\]

and hence \(\chi\) and \(\epsilon\) have the form

\[
\chi \equiv \theta \equiv \varepsilon L \cdot \chi - \chi
\]

Thus it follows that

So

\[
\theta = \varepsilon L \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \epsilon} = \varepsilon L \cdot \chi - \chi
\]

and using this, we have

\[
0 = \varepsilon L \frac{\partial}{\partial \theta} - \chi L \cdot \chi
\]

where \(\chi\) and \(\epsilon\) are still valid. But in this case \(\epsilon = 0\), so we have

\[
\theta = \varepsilon L \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \epsilon} = \varepsilon L \cdot \chi - \chi
\]
Thus, for the timelike case, the term will always contribute negatively to the right-hand side if
\[ T_{\mu\nu} - \frac{1}{2} T_{\lambda\lambda} g_{\mu\nu} \geq 0 \] (27)
for all timelike vectors, which is called the strong energy condition. For a perfect fluid,
the energy-momentum tensor is given by
\[ T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} + p g_{\mu\nu} \] (28)
where \( \rho \) is the energy density, \( p \) is the pressure, and \( U_{\mu} \) is the four-velocity of the fluid.

(11)

\[ 0 \geq d_{x} + \sigma \] and \[ 0 \geq d + \delta \]

Thus, for the timelike case, the term will always contribute negatively to the right-hand side if
\[ \epsilon (L \cdot \ell)(d + \sigma) = a_L a_{\ell} d + \delta L \]

(30)

\[ 0 \geq a_d a^{\ell} L \]

For the lightlike case, the term on the right-hand side of the Raychaudhuri equation will always contribute negatively provided
\[ T_{\mu\nu}^{\ell\ell} \geq 0 \] (30)
for all lightlike vectors, which is called the null energy condition. Show that for a

(13)

\[ 0 \geq d_{x} + \sigma \] and \[ 0 \geq d + \delta \]

perfect fluid, the null energy condition is equivalent to
\[ \sigma \geq \epsilon (L \cdot \ell) (d + \sigma) \] (30)
for all lightlike vectors, which is called the null energy condition. Show that for a

(31)

\[ 0 \geq d_{x} + \sigma \] and \[ 0 \geq d + \delta \]

perfect fluid, the null energy condition is equivalent to
\[ \epsilon (L \cdot \ell)(d + \sigma) = a_L a_{\ell} d + \delta L \]

(32)

\[ 0 \geq a_d a^{\ell} L \]

Thus, for the lightlike case, the term will always contribute negatively to the right-hand side if
\[ T_{\mu\nu}^{\ell\ell} \geq 0 \] (32)
for all lightlike vectors, which is called the null energy condition. For a perfect fluid,