PROBLEM SET 9 SOLUTIONS

DUE DATE: Thursday, April 19, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This is a short problem set with one problem about the Kruskal coordinate system, as discussed in lecture and in Chapter 5 of Carroll.

PROBLEM 1: KRUSKAL EXTENDED SCHWARZSCHILD SOLUTION (35 pts)

In the Kruskal coordinate system, the metric is given by

\[ ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} (-dT^2 + dR^2) + r^2 d\Omega^2, \]  

where the coordinates are \( T, R, \theta, \) and \( \phi, \) and \( r \) is defined implicitly by the relation

\[ T^2 - R^2 = \left( 1 - \frac{r^2}{2GM} \right) e^{r/2GM}. \]

As usual,

\[ d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2. \]

Within the Kruskal description, Eq. (2) defines \( r \) as a function of \( T \) and \( R, \) but the same equation also describes how \( T^2 - R^2 \) is related to the Schwarzschild coordinate \( r. \)

The Kruskal plane is divided into quadrants, as shown in Figure 1. Eqs. (1) and (2)

![Figure 1: The quadrants of the Kruskal diagram.](image)

define the Kruskal metric in all quadrants. We know that the metric defined in this way satisfies \( R_{\mu\nu} = 0 \) in quadrant I, but analyticity then implies that it satisfies \( R_{\mu\nu} = 0 \) everywhere, so it describes a full spacetime satisfying the vacuum Einstein equations.
The identification of the $r$ in Eqs. (1) and (2) with the Schwarzschild coordinate $r$ is also valid in all quadrants, since $r^2$ is the coefficient of $d\Omega^2$ in both coordinate systems.

In the first quadrant, $T$ and $R$ can be related to the Schwarzschild coordinates $t$ and $r$ by

$$T = \left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \sinh \left( \frac{t}{4GM} \right),$$

$$R = \left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \cosh \left( \frac{t}{4GM} \right),$$

which in turn implies that

$$\frac{T}{R} = \tanh \left( \frac{t}{4GM} \right).$$

in the first quadrant.

(a) [5 pts] Write the analogue of Eqs. (4) for quadrant II, the black hole interior. Your formulas should be consistent with Eqs. (1) and (2), and also with the standard form of the Schwarzschild metric for $t$, $r$, $\theta$, and $\phi$. The Schwarzschild coordinate $t(T, R)$ cannot be continuous throughout the $(T, R)$ plane, since $t = \pm \infty$ on the $45^\circ$ lines. Nonetheless, your answers to this part and the next should be consistent with the standard convention that $t(T, R)$ is defined to be as continuous as possible, in the sense that if $t(T, R)$ approaches $\pm \infty$ as a line is approached from one side, then it approaches $\pm \infty$ with the same sign as the line is approached from the other side. Hint: one way to proceed is to modify the derivation done in class (or in Carroll) to apply to the black hole interior, with $r < R_S$. Note that the equation for $r_*$ would need to be modified.

ANSWER:

Following the hint, we note that radial geodesics (i.e., geodesics with $\theta = \phi =$ constant) in Schwarzschild coordinates obey the equation

$$0 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 \implies \frac{dt}{dr} = \pm \frac{r}{R_S - r},$$

whether $r$ is greater than or less than $R_S \equiv 2GM$. The choice of sign ($\pm$) allows for both ingoing and outgoing trajectories. Choosing the $-$ sign,

$$dt = -\frac{r}{R_S - r} \, dr \implies t = -\int \frac{r}{R_S - r} \, dr = r + R_S \ln \left( \frac{R_S - r}{R_S} \right).$$

The above indefinite integral motivates the definition

$$r_* = r + R_S \ln \left( \frac{R_S - r}{R_S} \right)$$

(S.3)
for quadrant II. Note that \( r_* \) is real when \( r < R_S \), and that \( dt = \pm dr_* \) for radial light rays. I chose the \( - \) sign in Eq. (S.2) so that \( r_* \) is written as \(+r\) plus a logarithmic correction. In terms of \( r_* \), the metric can be written as

\[
\mathrm{d}s^2 = - \left( 1 - \frac{R_S}{r} \right) (\mathrm{d}t^2 - \mathrm{d}r_*^2) - r^2 \, \mathrm{d}\Omega^2 ,
\]

where of course

\[
\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2 .
\]

To express the factor \((1 - R_S/r)\) in terms of the other variables, we first introduce lightlike coordinates

\[
u \equiv t - r_* , \quad v \equiv t + r_* ,
\]

so

\[
\mathrm{d}s^2 = \left( \frac{R_S}{r} - 1 \right) \, \mathrm{d}u - r^2 \, \mathrm{d}\Omega^2 .
\]

By exponentiating Eq. (S.3), we find

\[
\frac{R_S - r}{R_S} = e^{(r_* - r)/R_S} ,
\]

which can be rewritten using Eq. (S.6) to give

\[
\left( \frac{R_S}{r} - 1 \right) = \frac{R_S}{r} e^{-(v-u)/(2R_S)} .
\]

The metric can then be rewritten as

\[
\mathrm{d}s^2 = \frac{R_S}{r} e^{-r/R_S} e^{(v-u)/(2R_S)} \, \mathrm{d}u - r^2 \, \mathrm{d}\Omega^2 .
\]

(It is not essential, but it is interesting to keep an eye on what happens with the Schwarzschild singularity at \( r = R_S \). The metric in Eq. (S.7) is still singular at \( r = R_S \), since the first factor vanishes, and Eq. (S.10) was obtained by merely rewriting this first factor in terms of \( u \) and \( v \). As \( r \to R_S \), one can see from Eq. (S.3) that \( r_* \to -\infty \), so \( (v-u) \to -\infty \), so the first term in Eq. (S.10) vanishes, as we expect from the derivation. But the singularity has now been pushed to infinity.)

To finish the transformation to the metric of Eqs. (1) and (2), it is useful to keep the definitions

\[
V \equiv T + R , \quad U \equiv T - R
\]

for all quadrants, so that \( U \) and \( V \) are a global relabeling of \( T \) and \( R \). But \( u \) and \( v \) were defined in terms of \( r_* \), and we found that the definition of \( r_* \) used in quadrant I becomes complex in quadrant II, so we adopted a new definition. So we should not be surprised if \( u \) and \( v \) have different meanings in quadrant II than they did
in quadrant I. In quadrant II, from the figure, we see that \( T \geq |R| \), so \( V \geq 0 \) and \( U \geq 0 \). Thus we adopt the definitions
\[
V \equiv e^{v/(2RS)} , \quad U \equiv e^{-u/(2RS)} ,
\]
where the equation for \( U \) has the opposite sign from the equation we used in quadrant I. Then
\[
e^{(v-u)/(2RS)} \, du \, dv = -4R_S^2 \, dU \, dV ,
\]
so
\[
ds^2 = -\frac{4R_S^3}{r} e^{-r/R_S} \, dU \, dV - r^2 \, d\Omega^2 
= \frac{4R_S^3}{r} e^{-r/R_S} (-dT^2 + dR^2) - r^2 \, d\Omega^2 ,
\]
as we wanted.

(To follow the disappearance of the Schwarzchild singularity, note that it disappears with Eq. (S.14). It is the fact that \( dV/\, dv \) and \( dU/\, du \) approach zero as \( r \to R_S \) that makes this possible. Note that as \( r \to R_S, \, u \to \infty \) and \( v \to -\infty \), so the exponents appearing in Eq. (S.12) vanish in this limit.)

To find how to relate \( r \) to \( T \) and \( R \), which is necessary to make use of Eq. (S.14), we can combine Eq. (S.9) with Eqs. (S.12) and (S.11) to find
\[
T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} ,
\]
which matches Eq. (2), as it should.

Finally, what we were really asked for was the expressions for the Kruskal coordinates \( T \) and \( R \) in terms of the Schwarzchild coordinates \( t \) and \( r \). This is exactly the transformation that we have constructed, but we did it in pieces. To give the final answer to the question, it is just a matter of combining the pieces:

\[
T = \frac{1}{2} (V + U) \\
= \frac{1}{2} \left[ e^{v/(2RS)} + e^{-u/(2RS)} \right] \\
= \frac{1}{2} \left[ e^{(t+r_*)/(2RS)} + e^{-(t-r_*)/(2RS)} \right] \\
= e^{r_*/(2RS)} \cosh \left( \frac{t}{2R_S} \right) ,
\]
so finally
\[
T = e^{r/(2RS)} \left(1 - \frac{r}{R_S}\right)^{1/2} \cosh \left( \frac{t}{2R_S} \right) .
\]
Similarly

\[ R = \frac{1}{2}(V - U) \]
\[ = \frac{1}{2}[e^{\nu/(2RS)} - e^{-u/(2RS)}] \]
\[ = \frac{1}{2}\left[ e^{(t+r_\ast)/(2RS)} - e^{-(t-r_\ast)/(2RS)} \right] \]
\[ = e^{r_\ast/(2RS)} \sinh\left( \frac{t}{2RS} \right), \]  
(S.18)

so finally

\[ R = e^{r/(2RS)} \left( 1 - \frac{r}{RS} \right)^{1/2} \sinh\left( \frac{t}{2RS} \right). \]  
(S.19)

To fully answer the question, we must verify that we have maintained continuity in \( t(T,R) \) at the boundary between quadrants I and II, in the sense described in the problem. In quadrant I, \( t \to \infty \) at this interface, so the same should be true for quadrant II. It is easy to see that it is, since Eqs. (S.17) and (S.19) imply that

\[ \frac{T}{R} = \coth\left( \frac{t}{2RS} \right), \]  
(S.20)

so \( T/R \to 1 \) implies that \( t \to \infty \).

An alternative approach to this problem might be to guess the answer, and then show that it has the desired properties. To fully answer the question, one would have to show that the transformation turns the Schwarzschild metric into the Kruskal metric, and one would have to also show that the time coordinate is continuous in the sense described in the problem.

(b) [5 pts] Write the analogue of Eqs. (4) for quadrants III and IV. In quadrant IV, does \( t \) increase in the upward or downward direction?

**ANSWER:**

We could repeat an analysis similar to the above, but there is no need to. For quadrant III, starting with the answer above, we can consider the transformation in which we simply reverse the signs of both \( R \) and \( T \):

\[ T = -e^{r/(2RS)} \left( 1 - \frac{r}{RS} \right)^{1/2} \cosh\left( \frac{t}{2RS} \right), \]
\[ R = -e^{r/(2RS)} \left( 1 - \frac{r}{RS} \right)^{1/2} \sinh\left( \frac{t}{2RS} \right). \]  
(S.21)
These transformations have the desired property of mapping the Schwarzschild region for \( r < R_S \) into quadrant III, and since the Kruskal metric is invariant under both \( R \rightarrow R' = -R \) and \( T \rightarrow T' = -T \), the metric for the new coordinates will be the desired Kruskal metric. The minus sign in the \( T \) equation is needed so that \((T,R)\) is in quadrant III. The minus sign in the \( R \) equation is not needed for \((T,R)\) to be in the correct quadrant, and it is not needed for the metric to be Kruskal. But it is needed to give the continuity in \( t \) as described. For quadrant I, the boundary with quadrant III is the line \( T = -R \), with \( R > 0 \), and \( t \rightarrow -\infty \) along this line. With both signs as shown in Eq. (S.21), this property holds also for quadrant III.

Similar reasoning shows that the transformation for quadrant IV can be constructed by starting with the transformations for quadrant I, and then reversing the signs for both \( T \) and \( R \):

\[
\begin{align*}
T &= -\left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \sinh \left( \frac{t}{4GM} \right) \\
R &= -\left( \frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \cosh \left( \frac{t}{4GM} \right).
\end{align*}
\]

Here the minus sign for \( R \) is needed to put \((T,R)\) in quadrant IV, and the minus sign in the \( T \) equation is needed for the continuity property in \( t \). The \( T \) equation then implies that \( t \) increases in the downward direction, as \( T \) decreases.

(c) [10 pts] Eq. (2) cannot be inverted analytically to give \( r \) as a function of \( T \) and \( R \), but it can easily be solved numerically. To 10 significant figures, what is the value of \( r/2GM \) corresponding the \( T^2 - R^2 = 0.5 \)? Construct a plot of \( r/2GM \) vs. \( T^2 - R^2 \) for \( T^2 - R^2 \) ranging from -5 up to its maximum value. (This should be a calculated plot, not a qualitative sketch. You may use any computer system that you find convenient.)

**ANSWER:**

Here we are just exploring Eq. (2). For \( T^2 - R^2 = 0.5 \), we must solve

\[
0.5 = (1 - x)e^x,
\]

where \( x = r/2GM \). Numerically, to 10 significant figures, \( x = 0.7680390470 \). This can be found, for example, by using Mathematica, with

\[
\text{NumberForm}[\text{FindRoot}[0.5 == (1 - x)E^x(x), \{x, 0.5\}], 16]
\]

which returns

\[
\{x \rightarrow 0.7680390470134656\}
\]

I used my own software to draw the graph, which should look like
Graph of $r/2GM$ vs. $T^2 - R^2$.

(d) [5 pts] Consider a particle that emerges from the white hole singularity at $(T, R) = (-1, 0)$, and then travels along the line $R = 0$ to $(T, R) = (1, 0)$, where it disappears into the black hole singularity. What is the total proper time $\tau$ experienced by this particle? (Hint: although the question was asked in the language of Kruskal coordinates, you may find it useful to think about Schwarzschild coordinates in answering it.)

ANSWER:

From the relation between Kruskal and Schwarzschild coordinates in quadrants II and III, we can see that the path can be described in Schwarzschild coordinates as twice the length of the path from $r = 0$ to $r = 2GM$, with $t = 0$ and $\theta$ and $\phi$ fixed. So

$$\tau = 2 \int_0^{RS} \left( \frac{RS}{r} - 1 \right)^{-1/2} dr = 2RS \int_0^1 \left( \frac{u}{1-u} \right)^{1/2} du , \quad (S.26)$$

where I changed the variable of integration to $u = r/RS$. Now changing the variable of integration to $u = \sin^2 \theta$, the integral becomes

$$\tau = 4RS \int_0^{\pi/2} \sin^2 \theta \ d\theta = \pi RS = 2\pi GM . \quad (S.27)$$

(The difficulty in using Kruskal coordinates for this problem is that $r$ appears in the metric, but it cannot be expressed as an explicit function of $T$ and $R$. However, one could formally set up the integral using Kruskal variables, and then one can change the variable of integration to $r$, and the integral reduces to Eq. [S.26] above.)

(e) [10 pts] If one describes the Kruskal spacetime in terms of slices of constant $T$, it appears as two separate spacetimes that are joined temporarily by a wormhole, or Einstein–Rosen bridge, as depicted in Carroll’s Figure 5.15, shown here as Figure [2]. At what value of $T$ does the Einstein–Rosen bridge appear? At what value of $T$ does it disappear? Let $r(\tau)$ denote the radius (i.e., the value of $r$) of the thinnest part of the...
Einstein–Rosen bridge as a function of a time variable $\tau$, where $\tau$ is the proper time as measured on a clock that appears and starts from $\tau = 0$ when the Einstein–Rosen bridge appears, and is always located at the smallest value of $r$, and at the same value of $\theta$ and $\phi$. Use numerical techniques to construct a plot of $r/2GM$ vs. $\tau/2GM$. (The hint from the previous part may also be useful here.)

**ANSWER:**

A diagram of the Kruskal coordinate space, showing its relation to the Schwarzschild coordinates in each quadrant, looks like the following:

![Diagram of the Kruskal coordinate space](image)

**Figure 3:** Diagram of the Kruskal $(T, R)$ coordinate space. Schwarzschild coordinates $(t, r)$ are shown, in units of $GM$. (The numbering of the quadrants does not match the numbering used by Carroll.)

The Einstein–Rosen bridge is the connection between the right and left-hand sides, along hypersurfaces of constant $T$. One can see from the diagram that such a bridge exists only for $-1 < T < 1$, so the bridge appears at $T = -1$ and
it disappears at $T = 1$. The radius at the thinnest part of the bridge is the value of $r$ at the left-right center of the diagram, i.e., at $R = 0$. The time variable $\tau$ that we are asked to use is the proper time of a clock that begins at $(T,R) = (-1,0)$, and moves along the $R = 0$ line. The proper time $\tau$ can be described more simply in Schwarzschild coordinates: for $T < 0$, it is the length of a trajectory at $t = 0$, extending in $r$ from $r = 0$ to some final value $r_f$. Using Schwarzschild coordinates to integrate, the length of this path is

$$\tau = \int_0^{r_f} \left( \frac{R_S}{r} - 1 \right)^{-1/2} dr .$$  \hspace{1cm} (S.28)

Using the same substitutions that were used in Eqs. (S.26) and (S.27), we find

$$\tau = 2R_S \int_0^{\theta_f} \sin^2 \theta \, d\theta ,$$ \hspace{1cm} (S.29)

where

$$\sin^2 \theta_f = \frac{r_f}{R_S} .$$ \hspace{1cm} (S.30)

Using the identity

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} ,$$ \hspace{1cm} (S.31)

the integral can be carried out, yielding

$$\tau = R_S[\theta_f - \sin \theta_f \cos \theta_f] .$$ \hspace{1cm} (S.32)

Finally, then, $\tau$ is related to $r_f$ by

$$\tau = R_S \sin^{-1}\left( \sqrt{\frac{r_f}{R_S}} \right) - \sqrt{r_f(R_S - r_f)} \quad \text{(for } T < 0) .$$ \hspace{1cm} (S.33)

The radius of the thinnest part of the Einstein–Rosen bridge is $r_f$, so the function that we are asked to plot, $r(\tau)$, is obtained by solving the above equation for $r_f(\tau)$. The equation cannot be inverted analytically, but it can be inverted numerically. For $T > 0$, $\tau$ is the proper time needed for $r$ to vary from 0 to $R_S$, which is $\pi R_S/2$, plus the proper time needed for $r$ to vary from $R_S$ to $r_f$:

$$\tau = \frac{1}{2} \pi R_S + \int_{r_f}^{R_S} \left( \frac{R_S}{r} - 1 \right)^{-1/2} dr$$

$$= \pi R_S - \int_0^{r_f} \left( \frac{R_S}{r} - 1 \right)^{-1/2} dr ,$$ \hspace{1cm} (S.34)
so

$$\tau = \pi R_S - \left[ R_S \sin^{-1}\left( \sqrt{r_f} / R_S \right) - \sqrt{r_f (R_S - r_f)} \right] \quad (\text{for } T > 0) . \quad (S.35)$$

The graph is then constructed by numerically inverting Eqs. (S.33) and (S.35), and it should look like

Graph of \( r/2GM \) vs. \( \tau/(2GM) \).

You were not asked to do so, but for comparison we can draw a graph of \( r/2GM \), where \( r \) is again the minimum radius of the Einstein–Rosen bridge, as a function of the Kruskal time coordinate \( T \):

Graph of \( r/2GM \) vs. \( T \).

Your graph should look like the graph of \( r/2GM \) vs. \( \tau/2GM \), and not like the graph immediately above.