

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.981 FALL 2009

PROBLEM SET 3

Post date: Monday, November 2nd

Due date: Tuesday, November 17th

1. **Monte-Carlo Integration:** Evaluate the integral

$$\int \int e^{-x \sin y} dx dy$$

over the interior of an ellipse whose boundary is given by

$$5x^2 - 6xy + 5y^2 = 2 .$$

Evaluate this with enough points to compute the integral with a variance smaller than 0.5%.

Before doing the integral, try to think up any tricks that may allow you to make the variance small with a minimum number of function evaluations. Hint: You can transform variables $(x, y) \rightarrow (u, v)$ such that the region of integration is simple. That will facilitate laying points out uniformly over the domain.

2. **Markov Chain Monte Carlo:** You are given a set of data which contains a signal of the following form,

$$s(t) = m(t; \beta_T, T_T) = A \sin \left[\beta_T (T_T - t)^{1/2} \right] ,$$

buried in white noise. The notation stresses that “ s ” is the actual signal you have measured, and “ m ” is a model. We assume that the model faithfully represents the signal (i.e., errors are statistical, rather than systematic), and that our task is to determine the “true” parameters for which m duplicates the signal s (modulo noise).

Somehow, you have learned that $A = 0.4$, exactly. Your goal is now to determine the two¹ remaining parameters β_T and T_T that characterize the signal. (The subscript T stands for “true,” and is added to differentiate the true values you are trying to estimate from the iterations i you will build in your MCMC chains.)

The data you are given² is uniformly spaced by $\Delta t = 10^{-4}$ seconds, and spans the domain $0 \leq t \leq 50$ seconds. From the fact that the signal’s frequency remains finite over the span of the data [note that the frequency $f(t) = d\text{Phase}/dt = (\beta_T/2)(T_T - t)^{-1/2}$ will diverge when $t \rightarrow T_T$], you know that $T_T > 50$ seconds; however, you also can see that the frequency does grow pretty quickly at the end, so T_T cannot be too large. You guess that $50 \text{ sec} \leq T_T \leq 55 \text{ sec}$. From physical constraints on the system that generated the signal, you determine that $18 \text{ sec}^{-1/2} \leq \beta \leq 22 \text{ sec}^{-1/2}$.

¹For the sake of the exercise, this is oversimplified from the kind of problem you would face in reality. At a minimum, the amplitude A should be determined, and there should be an overall phase offset.

²Downloadable from the course website.

By MCMC, estimate β_T and T_T . You will need the following ingredients to implement the Metropolis-Hastings algorithm:

- Use the following prior distribution functions for the iterations β_i and T_i that you draw:

$$\begin{aligned} p(T_i) &= 1 && 50 \text{ sec} \leq T_i \leq 55 \text{ sec} , \\ &= 0 && T_i < 50 \text{ sec}, T_i > 55 \text{ sec} ; \\ p(\beta_i) &= 1 && 18 \text{ sec}^{-1/2} \leq \beta_i \leq 22 \text{ sec}^{-1/2} , \\ &= 0 && \beta_i < 18 \text{ sec}^{-1/2}, \beta_i > 22 \text{ sec}^{-1/2} . \end{aligned}$$

- Use the following posterior distribution π to describe how well a model $m(\beta_i, T_i)$ fits the data s :

$$\pi[m(\beta_i, T_i), s] = p(\beta_i)p(T_i) \exp [-(1/2)\langle m(\beta_i, T_i)|s \rangle] ,$$

where the cross correlation is given by

$$\langle m(\beta_i, T_i)|s \rangle = \int_0^{50 \text{ sec}} dt [m(\beta_i, T_i) - s(t)]^2 .$$

- For your proposal distribution, use Gaussian deviates centered on the previous draw. In other words, pick $q = q_\beta q_T$, where

$$\begin{aligned} q_\beta(\beta_{i+1}|\beta_i) &= \frac{1}{2\pi\sigma_\beta} e^{-(\beta_{i+1}-\beta_i)^2/2\sigma_\beta^2} , \\ q_T(T_{i+1}|T_i) &= \frac{1}{2\pi\sigma_T} e^{-(T_{i+1}-T_i)^2/2\sigma_T^2} . \end{aligned}$$

In a real application, determining σ_T and σ_β would be a significant challenge. For the purpose of this exercise, I have found that $\sigma_\beta = 0.003 \text{ sec}^{-1/2}$, $\sigma_T = 0.003 \text{ sec}$ works well.) You may begin your chains with any value of β and T consistent with the prior distributions.

Build a chain with a length of $N = 300,000$. By plotting β_i and T_i for $1 \leq i \leq N$, determine the length of the burn-in, N_b , that was necessary before the chains appear to converge. For both β and T , estimate the true values β_T and T_T using the estimators

$$\begin{aligned} \beta_T &\simeq \langle \beta \rangle \pm \delta\beta , \\ T_T &\simeq \langle T \rangle \pm \delta T , \end{aligned}$$

where

$$\begin{aligned} \langle f \rangle &\simeq \frac{1}{N - N_b} \sum_{i=N_b}^N f_i , \\ (\delta f)^2 &\simeq \frac{1}{N - N_b} \sum_{i=N_b}^N (f_i - \langle f_i \rangle)^2 . \end{aligned}$$

(I suggest evaluating the error term as written here, rather than evaluating $\langle f^2 \rangle - \langle f \rangle^2$. For an average over hundreds of thousands of elements, I found that truncation error tends to badly skew the results with this second form.)