MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.981 FALL 2007

PROBLEM SET 4

Post date: Tuesday, Oct 23 2007
Due date: Tuesday, Oct 30 2007

A few simple problems to practice concepts we've looked at lately.

1. *Rotating our polarization state.* Our canonical GW propagates down the z-axis with $h_{xx} = -h_{yy} = h_+$, $h_{xy} = h_{yx} = h_\times$. This is well-adapted to a detector that is aligned with the x- and y-axes (at least for measuring h_+).

   Show that if the x- and y-axes are rotated by an angle ψ that one can define new polarization states, related to the old ones by

$$h_+^{\rm new} + i h_\times^{\rm new} = \left(h_+^{\rm old} + i h_\times^{\rm old}\right) e^{-2i\psi} \ .$$

   This is basically just an exercise in matrix multiplication. The tensor representating the GW is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

   We wish to rotate by an angle ψ about the z-axis. The rotation matrix that affects this rotation is

$$R_{\mu\nu}(\psi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & \sin\psi & 0 \\ 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

   The rotated representation is given by $h^{\rm new}_{\mu\nu} = h_{\alpha\beta} R^\alpha_\mu R^\beta_\nu$; in matrix notation (such as you might evaluate using e.g. Mathematica), this would be $\mathbf{h}^{\rm new} = \mathbf{R} \cdot \mathbf{h} \cdot \mathbf{R}^{\rm T}$, where $\mathbf{R}^{\rm T}$ is the transpose of the matrix $\mathbf{R}$. The result of this multiplication yields

$$\begin{aligned} h^{\rm new}_{xx} = -h^{\rm new}_{yy} \equiv h^{\rm new}_+ &= h_+ \cos 2\psi + h_\times \sin 2\psi \\ h^{\rm new}_{xy} = h^{\rm new}_{yx} \equiv h^{\rm new}_\times &= -h_+ \sin 2\psi + h_\times \cos 2\psi \ . \end{aligned}$$

   Gathering terms as suggested above, we find

$$\begin{aligned} h_+^{\rm new} + i h_\times^{\rm new} &= h_+ \left(\cos 2\psi - i \sin 2\psi\right) + h_\times \left(\sin 2\psi + i \cos 2\psi\right) \\ &= h_+ e^{-2i\psi} + i h_\times \left(\cos 2\psi - i \sin 2\psi\right) \\ &= (h_+ + i h_\times) e^{-2i\psi} \ . \end{aligned}$$

2. In class, I discussed the fluctuation-dissipation theorem for a mechanical system. Generalize my argument to an R-L-C circuit to argue that a current noise must exist in any finite temperature resistor. Compute the spectrum of this current noise (you may take the high temperature, classical limit — don't worry about quantum statistics!) Integrating over a bandwidth of 1 Hz, how much rms current (in nanoamps) do we expect in a room temperature resistor?

   For a series RLC circuit driven by some fluctuating voltage, the "equation of motion" can be written

$$L\ddot{q} + R\dot{q} + L\omega_0^2 q = \delta V$$

   where q **is charge and** $\omega_0^2 = 1/LC$**. (Once could also write this in terms of current, but then the capacitance term is a bit annoying.) The remainder of the problem follows in an identical manner to the calculation done in class, merely replacing** m **with** L **and** γ **with** R**.**

   From the rules we learned in lecture, we can easily relate the spectrum of q **to the spectrum of voltage fluctuations:**

$$\begin{aligned} S_q(f) &= \frac{S_{\delta V}(f)}{|L[\omega_0^2 - (2\pi f)^2] + 2\pi i R f|^2} \\ &= \frac{S_{\delta V}(f)}{\sqrt{L^2[\omega_0^2 - (2\pi f)^2]^2 + 4\pi^2 R^2 f^2}} . \end{aligned}$$

   From the basics of thermal physics, we can assume that each mode of the oscillator has a mean energy $\bar{E} = k_B T$**. We can also evaluate it as**

$$\begin{aligned} \bar{E} &= L\omega_0^2 \overline{q^2} \\ &= L\omega_0^2 \int_0^\infty S_q(f) df \\ &= \frac{L\omega_0^2 S_{\delta V}}{4L\omega_0^2 R} \end{aligned}$$

   so

$$S_{\delta V} = 4Rk_B T .$$

   The first line just uses the fact that the mean energy stored in the capacitor is $\overline{q^2}/(2C)$**, and that this is half the mean energy in the circuit. The rest is just definitions and math.**

   The rms current noise in a resistor is given by

$$\begin{aligned} \sigma_I &= \sqrt{S_{\delta V}\Delta f}/R \\ &= 2\sqrt{\frac{k_B T \Delta f}{R}} . \end{aligned}$$

   Plugging in some numbers, we have

$$\sigma_I = 0.012\,\mathrm{nanoamp} \left(\frac{\Delta f}{1\,\mathrm{Hz}}\right)^{1/2} \left(\frac{100\,\mathrm{Ohm}}{R}\right)^{1/2} \left(\frac{T}{293\,K}\right)^{1/2} .$$