1.1 Taylor Expansions

Problem 1.1: Construct the Taylor series to 3rd order for $\tan(x)$ and $\tanh(x)$ expanded around $x = 0$.

Problem 1.2: Calculate the Taylor expansion in two-dimensions of the functions

$$(a)\ f(x,y) = \frac{x^2}{xy+1}, \qquad (b)\ g(x,y) = e^{x\sin(y)} \tag{1.1.1}$$

around the point $(x,y) = (0,0)$.

Problem 1.3: Consider the function

$$f(x) = \frac{1}{1-x} \tag{1.1.2}$$

and write down the Taylor expansion around $x = 0$. Now define the truncated Taylor expansion

$$f_N(x) = \sum_{n=0}^{N} \frac{1}{n!}\frac{d^n f}{dx^n} x^n . \tag{1.1.3}$$

As a function of how many terms you keep in the truncation, plot $|f_N(x) - f(x)|$ at $x = \frac{1}{2}$ and at $x = \frac{3}{2}$. Comment on what you see.

Problem 1.4: Consider the function

$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases} \tag{1.1.4}$$

and investigate its Taylor series expanded around $x = 0$ by calculating its derivatives. Does the Taylor expansion converge for any x?

Problem 1.5: Solve the equation

$$\mathrm{e}^{y-1} = 1 - \epsilon y$$

for $y(\epsilon)$ to second order in the small parameter ϵ, using the ansatz $y(\epsilon) = y_0 + y_1\epsilon + \frac{1}{2!}y_2\epsilon^2 + \mathcal{O}(\epsilon^3)$. Use both of the following approaches.

1. Method 1: expansion of equation. Insert the ansatz for $y(\epsilon)$ into the given equation, Taylor expand each term to order $\mathcal{O}(\epsilon^2)$, and collect terms having the same power of ϵ to obtain an equation of the form $0 = \sum_n F_n \epsilon^i$. The coefficient of each ϵ^n must vanish, yielding a hierarchy of equations, $F_n = 0$. Starting from $n = 0$, solve these successively for the y_n, using knowledge of the previously determined $y_{i<n}$ at each step. [Check your results: $y_2 = 1$.

2. Method 2: repeated differentiation. Method 1 can be viewed from the following perspective: the given equation is written in the form $0 = \mathcal{F}(y(\epsilon), \epsilon) \equiv F(\epsilon)$, and the r.h.s. is brought into the form $\sum_n F_n \epsilon^n$. The latter process can be streamlined by realizing that $F_n = \frac{1}{n!} \mathrm{d}_\epsilon^n F(\epsilon)\big|_{\epsilon=0}$ Hence, the n th equation in the hierarchy, $F_n = 0$, can be set up by simply differentiating the given equation n times and then setting ϵ to zero, $0 = \mathrm{d}_\epsilon^n F(\epsilon)\big|_{\epsilon=0}$. Use this approach to find a hierarchy of equations for y_0, y_1 and y_2. Hint: since $F(\epsilon)$ depends on ϵ both directly and via $y(\epsilon)$, the chain rule must be used when computing derivates, e.g. $\mathrm{d}_\epsilon F(\epsilon) = \partial_y \mathcal{F}(y, \epsilon) y' + \partial_\in \mathcal{F}(y, \epsilon)$

3. Method 2 has the advantage that it systematically proceeds order by order: information from $\mathcal{O}(\epsilon^n)$ is generated at just the right time, namely when it is needed in step n for computing y_n. As a result, this method is often more convenient than method 1, particularly if the dependence of $\mathcal{F}(y, \epsilon)$ on y is nontrivial. Using this idea, use mathematica to compute $y(\epsilon)$ to 8th order in ϵ.

Problem 1.6: What order ODEs are the following? Are they linear or non-linear?

1. $\frac{d^3y}{dx^3} + \sin(x)\frac{d^2y}{dx^2} = x^3 y(x)$
2. $\frac{dx}{dt} - x^2(t) = 2t^2 x(t)$
3. $\frac{dx}{dt} = 2t^2 x(t)$

Problem 1.7: Check explicitly that the series in Eq. (1.1.15) solves $\dot{x} = -\gamma x$.

Problem 1.8: A second order ODE such as (1.1.17) requires two boundary conditions to solve. What happens if three boundary conditions (eg, $x(0)$, $x'(0)$, $x'(1)$) are specified?

Problem 1.9: Derive the expressions for the series expansions of the cosine ans sine functions and verify Eq. (1.1.18).

Problem 1.10: For a particle moving in a potential $V(x) = x^4 - x^3 - 24x^2 + 9x - 180 = (x-3)(x+3)(x-5)(x+4)$, describe the motion if the inital poistion is $x(0) = 1, 3, 7$.