1.2 First order ordinary differential equations

Problem 1.11: By thinking about finite difference approximations to derivatives, verify Eq. (1.2.3).

Problem 1.12: Solve the ODE $\ddot{x} = -\omega_0^2 x$ using the general method, assuming energy E, and starting from $x(t = 0) = x_0$.

Problem 1.13: Find the solution $x(t)$ for the position of a particle satisfying $\dot{x} = F(x)$ with $F(x) = \alpha x^2$.

Problem 1.14: Use a computer algebra program such as mathematica to solve the ODE subject to the given conditions

$$y'(x) = y(x)\cos(xy(x))), \qquad y(0) = 1$$

and plot the result for y(x) over the range $-5 \leq x \leq 5$.

Problem 1.15: Instead of logistic growth, with $G(N) = g_0 + g_1 N + g_2 N^2$ in Eq. (1.2.6), solve for the growth curve $N(t)$ in the case that $G(N) = g_1 N + g_4 N^4$.

Problem 1.16: Consider a population of excited atoms undergoing spontaneous decay that are replenished by a pumping mechanism at a rate that depends on the square of the number of atoms currently excited, such that the number of excited atoms at a given time, $N(t)$, satisfies

$$\frac{dN}{dt} = -\lambda N + \alpha N^2.$$

If there are initially N_0 excited atoms present, find $N(t)$ at later times, and comment on the results if λ/α is greater than, less than, or equal to N_0.

Problem 1.17: An object of mass m initially at rest at $z = z_0 > 0$ (positive z is up) falls under gravity in a resistive medium where the resistivity, η, is height dependent. The motion satisfies the equation

$$m\frac{dv}{dt} = \eta(z)v^2 - mg$$

where $\eta(z)$ is the coefficient of resistance at a distance z into the medium. [Upwards is taken as the positive z direction.]

1. Rewrite this equation such that the independent variable is z and the dependent variable is v.

2. Show that the resultant equation is inexact, and determine the integrating factor.
3. What is the velocity of the object as a function of distance if

$$\eta(z) = \frac{1}{2}\lambda(1 - \tanh[(\lambda/m)z])$$

4. What is the terminal velocity of the particle for η as given in part (c)?

Problem 1.18: Use the mathematica function `DSolve[eqn,y[x],x]` to find the generic form of solutions to the ODE $y'(x) = y^n(x)$ and verify your result.

Problem 1.19: An example with a fold bifurcation perhaps - not sure if this is too much of an extension of the problems in lectures ?

Consider the equation

$$\frac{dx}{dt} = f(x,c) = x(x-1) + c \tag{1.2.1}$$

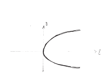
and analyze the lines of equilibrium in th $\{x, c\}$ plane.

Problem 1.20: The bifurcations we encountered correspond to points in parameter space of the ODE $\dot{y} = \phi(y)$ when two fixed points (solutions to $\phi(y^*) = 0$ collide or merge. This is required by the analytic continuity of the Taylor expansion of the function $\phi(y)$. The same continuity requires the alternation of stable and unstable fixed points observed in the above examples. For the same reason a fixed point cannot appear or disappear in isolation, but must do so by merging with another fixed point.

If we do not insist upon maintaining a stable fixed point as we have done so far, other forms of bifurcation are possible. For example, a mechanism by which a pair of fixed points disappear is provided by

$$\dot{y} = \epsilon - y^2\,, \tag{1.2.2}$$

where a pair of fixed points (one stable and one unstable), absent for $\epsilon < 0$, is created (at $\pm\sqrt{\epsilon}$) for $\epsilon > 0$.



Conversely for

$$\dot{y} = \epsilon + y^2 - y^3\,, \tag{1.2.3}$$

the pair of stable/unstable fixed points (with eigenvalues $\pm\sqrt{\epsilon}$) collide and disappear for $\epsilon > 0$ in a *fold bifurcation*. Note that to prevent divergence to infinity a stabilizing term $-y^3$ is added to the equation. For small ϵ, this leads to an additional stable fixed point at $y^* \approx 1$. Outcomes attracted to the stable fixed point at $-\sqrt{-\epsilon}$ for $\epsilon < 0$ now jumps discontinuously to $y^* \approx 1$ for $\epsilon > 0$.

1. Find and sketch the potentials that lead to the above ODEs via gradient descent.

2. Find the solutions $y(t)$ to the above equations starting from $y(t = 0) = y_0$.

Problem 1.21: Plot the solutions (1.2.10) and (1.2.14) for $r = \epsilon = 0.1, 0.01, 0.001$ and compare their large t behaviour to that in (1.2.16), $y(t) \sim t^{-1/(p-1)}$ for $p = 2, 3$ respectively.