1.4 General linear ordinary differential equations

Problem 1.32: Find values of k so that $y = \mathrm{e}^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

Problem 1.33: Given the equation $Ay'' + By' + Cy = 0$, construct the general solution $y(x) = ae^{kx}$ and write down the auxiliary equation that k must satisfy. If the roots of the auxiliary equation are complex (in this case, they will always be the complex conjugate of each other) and are denoted by $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$ show that the general solution is:

$$y(x) = \mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Problem 1.34: The Abraham-Lorentz equation for a classical charged particle feeling the effects of its own electric field is given by

$$m\left(\frac{d^2x}{dt^2} - \tau\frac{d^3x}{dt^3}\right) = F$$

where m is the mass of the particle, F is any externally applied force and

$$\tau = \frac{e^2}{6\pi\epsilon_0 mc^3}$$

is a constant with the units of time with the value, for an electron, of 6.26×10^{-24} sec (the time for light to cross the 'width' of the particle).

1. Find the general solution for this equation.
2. If the driving term is $F = qE_0\sin\omega t$ determine the full solution of this equation expressed in terms of the initial values $x(0), \dot{x}(0)$, and $\ddot{x}(0)$
3. What is unphysical about this solution? For what initial conditions would this unphysical behaviour not occur?

Problem 1.35: *Damping and initial conditions:* Consider a damped displacement $x(t)$, satisfying the homogeneous differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0.$$

1. Subject to the initial conditions $x(t=0)=x_0$ and $\dot{x}(t=0)=0$, find the solution $x(t)$ in the three cases where the motion is overdamped, critically damped, and underdamped. In each case, make a sketch of the solution.

2. Using the superposition principle, or otherwise, find the solutions with the initial condition $x(t=0)=x_0$ and $\dot{x}(t=0)=v_0$.

Problem 1.36: Express the following quantities in terms of the *quality factor* $Q=\omega_0/\gamma$.

1. The ratio of the steady-state amplitudes between oscillators driven at resonance ($\omega=\omega_0$), and with a uniform force ($\omega=0$), with the same force amplitude.

2. The fraction of energy dissipated per cycle in a free oscillatory decay.

Problem 1.37: Given a second order inhomgeneous ODE

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

a *particular integral* is a function, $y_{\mathrm{p}}(x)$, which satisfies the equation. The full solution to the ODE is given by the combination of the general solution to the corresponding homogeneous ODE (where the RHS is set to 0) and a particular integral. Verify that the following pairs of $f(x)$'s and trial solutions solve the ODE

	$f(x)$	Trial solution
(1)	constant term c	constant term k
(2)	linear, $ax+b$	$Ax+B$
(3)	polynomial in x	polynomial in x
	of degree r :	of degree r :
	$ax^r+\cdots+bx+c$	$Ax^r+\cdots+Bx+k$
(4)	$a\cos kx$	$A\cos kx+B\sin kx$
(5)	$a\sin kx$	$A\cos kx+B\sin kx$
(6)	ae^{kx}	$A\mathrm{e}^{kx}$
(7)	$a\mathrm{e}^{-kx}$	$A\mathrm{e}^{-kx}$

Problem 1.38: *A simple seismometer* consists of a mass m hanging from a spring of Hookian constant $m\omega_0^2$. The other end of the spring is connected to a rigid frame attached to, and vibrating with, the ground. The mass also experiences a drag force equal to $\gamma m v_r$, where v_r is its velocity relative to the air, which is assumed to move with the ground. The recorded signal $s(t)$ is the vertical displacement of the mass *relative to the frame*, i.e. the length of the spring.

1. Write down the equation of motion for the displacement $x(t)$ of the mass.

2. If the ground vibrates with a vertical displacement $H\cos(\omega t)$, find the steady–state solution for $x(t)$.

3. Find the amplitude A of the steady–state signal $s(t)$ recorded by the seismometer, and show that *when critically damped*, it is given by

$$\frac{A}{H} = \frac{\omega^2}{\omega^2 + \omega_0^2}.$$

4. How will the answer change if in calculating the drag force, we assume that air is stationary, and does not move with the ground?

Problem 1.39: *Damped forcing:* In lectures, we used complex exponentials to calculate the steady–state response to harmonic forcing. This method can in fact be generalized to cases where the exponent is itself complex, of the form $e^{(\alpha+i\beta)t}$. Use this observation to answer the following questions about a non-harmonically driven oscillator with the equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f e^{\alpha t}\cos(\beta t).$$

1. Find the analog of the steady–state solution, indicating its amplitude and phase relative to the forcing function.

2. Give the simplified answer in the case $\alpha = -\gamma/2$ and $\beta = \omega_0$. Describe why strong beating is expected in this case, once the initial conditions are properly taken into account. Find the beat frequency for large $Q = \omega_0/\gamma$.

3. What is the solution in the limit of $\alpha = -\gamma/2$ and $\beta = \sqrt{\omega_0^2 - \gamma^2/4}$?

Problem 1.40: *Average Energy and Power:* Consider the steady–state solution for a damped oscillating mass m, driven by a force $F = F_\omega \cos(\omega t)$.

1. Write down the net energy of the system (mass and restoring spring) as $E(t) = \overline{E} + \Delta\cos(\beta t + \phi)$, and identify $\overline{E}$, Δ, β and ϕ.

2. Show that the average energy $\overline{E}$ is mostly kinetic for $\omega \gg \omega_0$, and mostly potential for $\omega \ll \omega_0$.

3. Calculate the ratio $\overline{P}_{loss}/\overline{E}$, where $\overline{P}_{loss}$ is the energy lost to dissipation.

Problem 1.41: *Parallel circuit:* A circuit consists of a capacitance C, a resistance R, and an inductance L, and a generator, connected in parallel. The generator produces a voltage $V(t) = V_0 \cos(\omega t)$.

1. Calculate the complex impedance, $Z = Z_R + iZ_I$, of the circuit, giving explicit expressions for Z_R and Z_I.
2. Why is the resonance condition obtained by setting $Z_I = 0$?
3. What is the mean power absorbed at resonance? Will it change if the capacitor and inductor are suddenly disconnected from the circuit?

Problem 1.42: *Overshooting during a Transient:* A lightly damped ($Q \gg 1$) system, initially at rest, is set into vibration by a harmonic driving force whose frequency is 1% higher than its natural resonance frequency. Estimate the maximum Q–value the system may have, if its amplitude during the initial build up is not to exceed its steady–state value by more than 10%.

Problem 1.43: *Critically damped transients:* A critically damped oscillator ($\omega_0 = \gamma/2$) is set into motion by a driving force $f_\omega \cos(\omega t)$.

1. Find the steady–state amplitude A, and phase ϕ.
2. If the oscillator starts at rest at $t = 0$, find the exact expression for the subsequent displacement $x(t)$.
3. Simplify the previous result for $\omega = \omega_0$.