# Chapter 1

# One variable

## 1.1 Taylor Expansions

**Problem 1.1:** Construct the Taylor series to 3rd order for $\tan(x)$ and $\tanh(x)$ expanded around $x = 0$.

**Problem 1.2:** Calculate the Taylor expansion in two-dimensions of the functions

$$(a)\ f(x,y) = \frac{x^2}{xy+1}, \qquad (b)\ g(x,y) = e^{x\sin(y)} \tag{1.1.1}$$

around the point $(x,y) = (0,0)$.

**Problem 1.3:** Consider the function

$$f(x) = \frac{1}{1-x} \tag{1.1.2}$$

and write down the Taylor expansion around $x = 0$. Now define the truncated Taylor expansion

$$f_N(x) = \sum_{n=0}^{N} \frac{1}{n!}\frac{d^n f}{dx^n} x^n \,. \tag{1.1.3}$$

As a function of how many terms you keep in the truncation, plot $|f_N(x) - f(x)|$ at $x = \frac{1}{2}$ and at $x = \frac{3}{2}$ . Comment on what you see.

**Problem 1.4:** Consider the function

$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases} \tag{1.1.4}$$

and investigate its Taylor series expanded around $x = 0$ by calculating its derivatives. Does the Taylor expansion converge for any $x$?

**Problem 1.5:** Solve the equation

$$\mathrm{e}^{y-1} = 1 - \epsilon y$$

for $y(\epsilon)$ to second order in the small parameter $\epsilon$, using the ansatz $y(\epsilon) = y_0 + y_1\epsilon + \frac{1}{2!}y_2\epsilon^2 + \mathcal{O}(\epsilon^3)$ . Use both of the following approaches.

1. Expansion of equation. Insert the ansatz for $y(\epsilon)$ into the given equation, Taylor expand each term to order $\mathcal{O}(\epsilon^2)$ , and collect terms having the same power of $\epsilon$ to obtain an equation of the form $0 = \sum_n F_n \epsilon^i$. The coefficient of each $\epsilon^n$ must vanish, yielding a hierarchy of equations, $F_n = 0$. Starting from $n = 0$, solve these successively for the $y_n$, using knowledge of the previously determined $y_{i<n}$ at each step. [Check your results: $y_2 = 1$].

2. Rpeated differentiation. Method 1 can be viewed from the following perspective: the given equation is written in the form $0 = \mathcal{F}(y(\epsilon), \epsilon) \equiv F(\epsilon)$, and the r.h.s. is brought into the form $\sum_n F_n \epsilon^n$. The latter process can be streamlined by realizing that $F_n = \frac{1}{n!} \mathrm{d}^n F(\epsilon)/d\epsilon^n \big|_{\epsilon=0}$ Hence, the $n$ th equation in the hierarchy, $F_n = 0$, can be set up by simply differentiating the given equation $n$ times and then setting $\epsilon$ to zero, $0 = \mathrm{d}^n F(\epsilon)/d\epsilon^n|_{\epsilon=0}$. Use this approach to find a hierarchy of equations for $y_0, y_1$ and $y_2$. Hint: since $F(\epsilon)$ depends on $\epsilon$ both directly and via $y(\epsilon)$, the chain rule must be used when computing derivates, e.g. $\mathrm{d}_\epsilon F(\epsilon) = \partial_y \mathcal{F}(y, \epsilon) y' + \partial_\in \mathcal{F}(y, \epsilon)$

3. The second method has the advantage that it systematically proceeds order by order: information from $\mathcal{O}(\epsilon^n)$ is generated at just the right time, namely when it is needed in step $n$ for computing $y_n$. As a result, this method is often more convenient than method 1, particularly if the dependence of $\mathcal{F}(y, \epsilon)$ on $y$ is nontrivial. Using this idea, use mathematica to compute $y(\epsilon)$ to 8th order in $\epsilon$.

**Problem 1.6:** What order ODEs are the following? Are they linear or non-linear?

1. $\frac{d^3y}{dx^3} + \sin(x)\frac{d^2y}{dx^2} = x^3 y(x)$
2. $\frac{dx}{dt} - x^2(t) = 2t^2 x(t)$
3. $\frac{dx}{dt} = 2t^2 x(t)$

**Problem 1.7:** Check explicitly that the series in Eq. (1.1.15) solves $\dot{x} = -\gamma x$.

**Problem 1.8:** A second order ODE such as (1.1.17) requires two boundary conditions to solve. What happens if three boundary conditions (eg, $x(0)$, $x'(0)$, $x'(1)$) are specified?

**Problem 1.9:** Derive the expressions for the series expansions of the cosine and sine functions and verify Eq. (1.1.18).

**Problem 1.10:** For a particle moving in a potential $V(x) = x^4 - x^3 - 24x^2 + 9x - 180 = (x-3)(x+3)(x-5)(x+4)$, describe the motion if the inital poistion is $x(0) = 1$, 3, 7.

## 1.2 First order ordinary differential equations

**Problem 1.11:** By thinking about finite difference approximations to derivatives, verify Eq. (1.2.3).

**Problem 1.12:** Solve the ODE $\ddot{x} = -\omega_0^2 x$ using the general method, assuming energy $E$, and starting from $x(t = 0) = x_0$.

**Problem 1.13:** Find the solution $x(t)$ for the position of a particle satisfying $\dot{x} = F(x)$ with $F(x) = \alpha x^2$.

**Problem 1.14:** Use a computer algebra program such as mathematica to solve the ODE numerically subject to the given conditions

$$y'(x) = y(x)\cos(xy(x))), \qquad y(0) = 1$$

and plot the result for y(x) over the range $-5 \leq x \leq 5$.

**Problem 1.15:** Instead of logistic growth, with $G(N) = g_0 + g_1 N + g_2 N^2$ in Eq. (1.2.6), solve for the growth curve $N(t)$ in the case that $G(N) = g_1 N + g_4 N^4$.

**Problem 1.16:** Consider a population of excited atoms undergoing spontaneous decay that are replenished by a pumping mechanism at a rate that depends on the square of the number of atoms currently excited, such that the number of excited atoms at a given time, $N(t)$, satisfies

$$\frac{dN}{dt} = -\lambda N + \alpha N^2.$$

If there are initially $N_0$ excited atoms present, find $N(t)$ at later times, and comment on the results if $\lambda/\alpha$ is greater than, less than, or equal to $N_0$.

**Problem 1.17:** An object of mass $m$ initially at rest at $z = z_0 > 0$ (positive $z$ is up) falls under gravity in a resistive medium where the resistivity, $\eta$, is height dependent. The motion satisfies the equation

$$m\frac{dv}{dt} = \eta(z)v^2 - mg$$

where $\eta(z)$ is the coefficient of resistance at a distance $z$ into the medium.

1. Rewrite this equation such that the independent variable is $z$ and the dependent variable is $v$.

2. Show that the resultant equation is inexact, and determine the integrating factor.
3. What is the velocity of the object as a function of distance if

$$\eta(z) = \frac{1}{2}\lambda(1 - \tanh[(\lambda/m)z])$$

4. What is the terminal velocity of the particle for $\eta$ as given in part (c)?

**Problem 1.18:** Use the mathematica function `DSolve[eqn,y[x],x]` to find the generic form of solutions to the ODE $y'(x) = y^n(x)$ and verify your result.

**Problem 1.19:** An example with a fold bifurcation perhaps - not sure if this is too much of an extension of the problems in lectures ?

Consider the equation

$$\frac{dx}{dt} = f(x, c) = x(x-1) + c \tag{1.2.1}$$

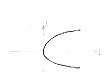
and analyze the lines of equilibrium in th $\{x, c\}$ plane.

**Problem 1.20:** The bifurcations we encountered correspond to points in parameter space of the ODE $\dot{y} = \phi(y)$ when two fixed points (solutions to $\phi(y^*) = 0$) collide or merge. This is required by the analytic continuity of the Taylor expansion of the function $\phi(y)$. The same continuity requires the alternation of stable and unstable fixed points observed in the above examples. For the same reason a fixed point cannot appear or disappear in isolation, but must do so by merging with another fixed point.

If we do not insist upon maintaining a stable fixed point as we have done so far, other forms of bifurcation are possible. For example, a mechanism by which a pair of fixed points disappear is provided by

$$\dot{y} = \epsilon - y^2\,, \tag{1.2.2}$$

where a pair of fixed points (one stable and one unstable), absent for $\epsilon < 0$, is created (at $\pm\sqrt{\epsilon}$) for $\epsilon > 0$.



Conversely for

$$\dot{y} = \epsilon + y^2 - y^3\,, \tag{1.2.3}$$

the pair of stable/unstable fixed points (with eigenvalues $\pm\sqrt{\epsilon}$) collide and disappear for $\epsilon > 0$ in a *fold bifurcation*. Note that to prevent divergence to infinity a stabilizing term $-y^3$ is added to the equation. For small $\epsilon$, this leads to an additional stable fixed point at $y^* \approx 1$. Outcomes attracted to the stable fixed point at $-\sqrt{-\epsilon}$ for $\epsilon < 0$ now jumps discontinuously to $y^* \approx 1$ for $\epsilon > 0$.

1. Find and sketch the potentials that lead to the above ODEs via gradient descent.

2. Find the solutions $y(t)$ to the above equations starting from $y(t = 0) = y_0$.

**Problem 1.21:** Plot the solutions (1.2.10) and (1.2.14) for $r = \epsilon = 0.1, 0.01, 0.001$ and compare their large $t$ behaviour to that in (1.2.16), $y(t) \sim t^{-1/(p-1)}$ for $p = 2, 3$ respectively.

## 1.3 Second order ordinary differential equations

**Problem 1.22:** Find values of $k$ so that $y = \mathrm{e}^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

**Problem 1.23:** Find the general solution of: $\frac{d^2y}{dx^2} + 4y = 0$

**Problem 1.24:** Given $ay'' + by' + cy = 0$, write down the auxiliary equation. If the roots of the auxiliary equation are complex (one root will always be the complex conjugate of the other) and are denoted by $k_1 = \alpha + \beta\mathrm{i}$ and $k_2 = \alpha - \beta\mathrm{i}$ show that the general solution is:

$$y(x) = \mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

**Problem 1.25:** Find the auxiliary equation for the differential equation $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$ Hence write down the general solution.

**Problem 1.26:** Use mathematica to solve the ODE subject to the given conditions

$$y''(x) = y'(x)\cos(xy(x))), \qquad y(0) = 1, y'(1) = 0$$

and plot the result for y(x) over the range $-5 \leq x \leq 5$.

**Problem 1.27:**

1. Write down the general solution for the differential equation

$$\frac{d^2y}{dt^2} + y = \cos rt$$

   and by substituting a suitable trial function, determine a particular integral for $r \neq 1$.

2. Express the complete solution to this equation in terms of the initial values $y(0)$ and $\dot{y}(0)$

3. By taking the limit of $r \to 1$ of the complete solution, determine the solution of this equation for $r = 1$.

4. Show that the same result follows if a trial solution $y_p = t(A\cos t + B\sin t)$ is used to determine the particular integral.

**Problem 1.28:** Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (1+x)e^{-x}; \quad y(0) = 0, \quad y'(0) = 1$$

by solving first for the general solution of the homogenous equation, and looking for an appropriate trial function to obtain a particular integral.

**Problem 1.29:** Show Eq. (1.3.6). What are the initial position and velocity of the solution in (1.3.6)?

**Problem 1.30:** Using the complex exponential, verify the following trigonometric identities:

1. $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
2. $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$
3. $\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$

**Problem 1.31:** Generalise the discussion of beats to the case where one tuning fork has twice the amplitude of the other.

**Problem 1.32:** Plot and discuss the position, velocity and acceleration of the solution $x = \tanh(t)$ in (1.3.18).

## 1.4 General linear ordinary differential equations

**Problem 1.33:** Find values of $k$ so that $y = \mathrm{e}^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Hence state the general solution.

**Problem 1.34:** Given the equation $Ay'' + By' + Cy = 0$, construct the general solution $y(x) = ae^{kx}$ and write down the auxiliary equation that $k$ must satisfy. If the roots of the auxiliary equation are complex (in this case, they will always be the complex conjugate of each other) and are denoted by $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$ show that the general solution is:

$$y(x) = \mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

**Problem 1.35:** The Abraham-Lorentz equation for a classical charged particle feeling the effects of its own electric field is given by

$$m\left(\frac{d^2x}{dt^2} - \tau\frac{d^3x}{dt^3}\right) = F$$

where $m$ is the mass of the particle, $F$ is any externally applied force and

$$\tau = \frac{e^2}{6\pi\epsilon_0 mc^3}$$

is a constant with the units of time with the value, for an electron, of $6.26 \times 10^{-24}$ sec (the time for light to cross the 'width' of the particle).

1. Find the general solution for this equation.
2. If the driving term is $F = qE_0\sin\omega t$ determine the full solution of this equation expressed in terms of the initial values $x(0), \dot{x}(0)$, and $\ddot{x}(0)$
3. What is unphysical about this solution? For what initial conditions would this unphysical behaviour not occur?

**Problem 1.36:** *Damping and initial conditions:* Consider a damped displacement $x(t)$, satisfying the homogeneous differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0.$$

1. Subject to the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = 0$, find the solution $x(t)$ in the three cases where the motion is overdamped, critically damped, and underdamped. In each case, make a sketch of the solution.

2. Using the superposition principle, or otherwise, find the solutions with the initial condition $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$.

**Problem 1.37:** Express the following quantities in terms of the *quality factor* $Q = \omega_0/\gamma$.

1. The ratio of the steady-state amplitudes between oscillators driven at resonance ($\omega = \omega_0$), and with a uniform force ($\omega = 0$), with the same force amplitude.

2. The fraction of energy dissipated per cycle in a free oscillatory decay.

**Problem 1.38:** Given a second order inhomgeneous ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

a *particular integral* is a function, $y_{\rm p}(x)$, which satisfies the equation. The full solution to the ODE is given by the combination of the general solution to the corresponding homogeneous ODE (where the RHS is set to 0) and a particular integral. Verify that the following pairs of $f(x)$'s and trial solutions solve the ODE

| | $f(x)$ | Trial solution |
|---|---|---|
| (1) | constant term $c$ | constant term $k$ |
| (2) | linear, $ax + b$ | $Ax + B$ |
| (3) | polynomial in $x$ of degree $r$ : $ax^r + \cdots + bx + c$ | polynomial in $x$ of degree $r$ : $Ax^r + \cdots + Bx + k$ |
| (4) | $a\cos kx$ | $A\cos kx + B\sin kx$ |
| (5) | $a\sin kx$ | $A\cos kx + B\sin kx$ |
| (6) | $a\mathrm{e}^{kx}$ | $A\mathrm{e}^{kx}$ |
| (7) | $a\mathrm{e}^{-kx}$ | $A\mathrm{e}^{-kx}$ |

**Problem 1.39:** *A simple seismometer* consists of a mass $m$ hanging from a spring of Hookian constant $m\omega_0^2$. The other end of the spring is connected to a rigid frame attached to, and vibrating with, the ground. The mass also experiences a drag force equal to $\gamma m v_r$, where $v_r$ is its velocity relative to the air, which is assumed to move with the ground. The recorded signal $s(t)$ is the vertical displacement of the mass *relative to the frame*, i.e. the length of the spring.

1. Write down the equation of motion for the displacement $x(t)$ of the mass.

2. If the ground vibrates with a vertical displacement $H\cos(\omega t)$, find the steady–state solution for $x(t)$.

3. Find the amplitude $A$ of the steady–state signal $s(t)$ recorded by the seismometer, and show that *when critically damped*, it is given by

$$\frac{A}{H} = \frac{\omega^2}{\omega^2 + \omega_0^2}.$$

4. How will the answer change if in calculating the drag force, we assume that air is stationary, and does not move with the ground?

**Problem 1.40:** *Damped forcing:* The method of complex exponentials can in fact be generalized to cases where the exponent is itself complex, of the form $e^{(\alpha+i\beta)t}$. Use this observation to answer the following questions about a non-harmonically driven oscillator with the equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f e^{\alpha t}\cos(\beta t).$$

1. Find the analog of the steady–state solution, indicating its amplitude and phase relative to the forcing function.

2. Give the simplified answer in the case $\alpha = -\gamma/2$ and $\beta = \omega_0$. Describe why strong beating is expected in this case, once the initial conditions are properly taken into account. Find the beat frequency for large $Q = \omega_0/\gamma$.

3. What is the solution in the limit of $\alpha = -\gamma/2$ and $\beta = \sqrt{\omega_0^2 - \gamma^2/4}$?

**Problem 1.41:** *Parallel circuit:* A circuit consists of a capacitance $C$, a resistance $R$, and an inductance $L$, and a generator, connected in parallel. The generator produces a voltage $V(t) = V_0\cos(\omega t)$.

1. Calculate the complex impedance, $Z = Z_R + iZ_I$, of the circuit, giving explicit expressions for $Z_R$ and $Z_I$.

2. Why is the resonance condition obtained by setting $Z_I = 0$?

3. What is the mean power absorbed at resonance? Will it change if the capacitor and inductor are suddenly disconnected from the circuit?

**Problem 1.42:** *Overshooting during a Transient:* A lightly damped ($Q \gg 1$) system, initially at rest, is set into vibration by a harmonic driving force whose frequency is 1% higher than its natural resonance frequency. Estimate the maximum $Q$–value the system may have, if its amplitude during the initial build up is not to exceed its steady–state value by more than 10%.

**Problem 1.43:** *Critically damped transients:* A critically damped oscillator ($\omega_0 = \gamma/2$) is set into motion by a driving force $f_\omega \cos(\omega t)$.

1. Find the steady–state amplitude $A$, and phase $\phi$.
2. If the oscillator starts at rest at $t = 0$, find the exact expression for the subsequent displacement $x(t)$.
3. Simplify the previous result for $\omega = \omega_0$.