2.1 Two variables

Problem 2.1: Find the general solution to the system of differential equations

$$\frac{dx_1}{dt} = x_1 + x_2$$
$$\frac{dx_2}{dt} = 4x_1 + x_2,$$

that is, the homogeneous linear system of ODEs with constant coefficients

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

1. Compute the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$.

2. Use these eigenvalues and eigenvectors to form the change of basis matrix $\mathbf{O}$ and the diagonal matrix $\mathbf{D}$ such that
$$\mathbf{D} = \mathbf{O}^{-1}\mathbf{A}\mathbf{O}$$

3. Write down the general solution of the diagonalised system in which the transformed degrees of freedom $\mathbf{z} = \mathbf{C}^{-1}\mathbf{x}$ have decoupled
$$\frac{d\mathbf{z}}{dt} = \mathbf{D}\mathbf{z} \quad \Longrightarrow \quad \mathbf{z} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$
where the λ_i are the eigenvalues of $\mathbf{A}$.

4. Finally, construct the solution of the original (coupled) system as
$$\mathbf{x} = \mathbf{O}\mathbf{z}.$$

Problem 2.2: Solve the following system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Plot trajectories of the two solutions in the x_1–x_2 plane using the `ParametricPlot[]` function in mathematica.

Problem 2.3: Solve the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that this has only a single eigenvector so a second solution must be found.

Problem 2.4: The equation of motion for a system of a spring and mass with damping is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m, c, k are the mass, damping coefficient and spring force constant respectively, and all are positive constants.

1. Write this equation as a system of two first order equations for $u_1(t) = x(t)$ and $u_2(t) = \frac{du_1}{dt}$
2. Show that $u_1 = 0, u_2 = 0$ is a *critical point* where $\frac{du_1}{dt} = \frac{du_2}{dt} = 0$ and analyze the nature and stability of the critical point as a function of the parameters m, c and k.