## 2.2 Multiple variables

**Problem 2.5:** Show that the $n$th order differential equation

$$\frac{d^n x}{dt^n} = c_0 x + c_1 \frac{dx}{dt} + \cdots + c_{n-1} \frac{d^{n-1}x}{dt^{n-1}}, \quad c_j \in \mathbb{R}$$

can be written as a system of first order differential equation.

**Problem 2.6:** According to the Lorentz force law, the motion of a charge $q$ in an electromagnetic field is given by

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where $m$ denotes the mass and $\mathbf{v}$ the velocity (a 3-vector). Assume that

$$\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

are constant fields. Find the solution of the initial value problem $\mathbf{v}(t = 0) = (0, 0, 0)$.

**Problem 2.7:** Express the dot-product and cross-product in terms of the summation convention. For the latter, introduce the Levi-Civita temsor $\epsilon_{ijk}$ for $\{i, j, k\} \in \{1, 2, 3\}$ which is totally antisymmetric under interchange of its indices ($\epsilon_{jik} = \epsilon ikj = -\epsilon_{kij} = -\epsilon_{ijk}$) and has $\epsilon 123 = 1$.

**Problem 2.8:** Show that

$$\begin{aligned}
\epsilon^{jik}\epsilon_{lmn} &= \delta^j_l \delta^i_m \delta^k_n + \delta^j_m \delta^i_n \delta^k_l + \delta^j_n \delta^i_l \delta^k_m - \delta^j_l \delta^i_n \delta^k_m - \delta^j_n \delta^i_m \delta^k_l - \delta^j_m \delta^i_l \delta^k_n, \\
\epsilon^{ijk}\epsilon_{lmk} &= \delta^i_l \delta^j_m - \delta^i_m \delta^j_l, \\
\epsilon^{ijk}\epsilon_{ljk} &= 2\delta^i_l, \\
\epsilon^{ijk}\epsilon_{ijk} &= 6
\end{aligned} .$$

To do so, discuss the antisymmetry properties of the first equation and then use that equation to derive the rest.

**Problem 2.9:** Use index notation to prove the identities

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\end{aligned}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are 3-component vectors.

**Problem 2.10:** Consider the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$$

Find the eigenvalues and eigenvectors.

**Problem 2.11:** Let $x_1, x_2, x_3 \in \mathbb{R}$. Find the eigenvalues of the $2 \times 2$ matrix

$$\begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}.$$

**Problem 2.12:** Let $\alpha, \beta, \gamma \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the $4 \times 4$ matrices

$$\begin{pmatrix} 0 & \cos(\alpha) & \cos(\beta) & \cos(\gamma) \\ \cos(\alpha) & 0 & 0 & 0 \\ \cos(\beta) & 0 & 0 & 0 \\ \cos(\gamma) & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \cosh(\alpha) & 0 & 0 & \sinh(\alpha) \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sinh(\alpha) & 0 & 0 & \cosh(\alpha) \end{pmatrix}.$$

**Problem 2.13:** Let $\alpha, \beta \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the $3 \times 3$ matrix

$$\begin{pmatrix} \alpha + \beta & 0 & \alpha \\ 0 & \alpha + \beta & 0 \\ \alpha & 0 & \alpha + \beta \end{pmatrix}$$

**Problem 2.14:** Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ with $A^2 = rA$, where $r \in \mathbb{C}$ and $r \neq 0$

1. Calculate $e^{zA}$, where $z \in \mathbb{C}$.
2. Let $U(z) = e^{zA}$. Let $z, z' \in \mathbb{C}$. Calculate $U(z)U(z')$

**Problem 2.15:** Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. We define $\sin(A)$ as

$$\sin(A) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} A^{2j+1}$$

If possible, find a $2 \times 2$ matrix $B$ over the real numbers $\mathbb{R}$ such that

$$\sin(B) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}?$$

**Problem 2.16:** Let

$$A = \begin{pmatrix} 2 & 3 \\ 7 & -2 \end{pmatrix}$$

Calculate $\det e^A$

**Problem 2.17:** For every positive definite matrix $A$, there is a unique positive definite matrix $Q$ such that $Q^2 = A$. The matrix $Q$ is called the square root of $A$.

1. Find the square root of the matrix

$$B = \frac{1}{2} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

2. Find the square root of the positive definite $2 \times 2$ matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

**Problem 2.18:** Let $A, B$ be $n \times n$ matrices and $t \in \mathbb{R}$. Show that

$$e^{t(A+B)} - e^{tA}e^{tB} = \frac{t^2}{2}(BA - AB) + \text{ higher order terms in } t.$$