2.4 Symmetries in matrices

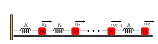
Problem 2.21: *Periodic chain in a harmonic trap:* Find eigenvalues and eigenvectors of a periodic chain confined in a harmonic trap, i.e. subject to the potential

$$V\left(x_1,\cdots x_N\right)=\frac{K}{2}\left[(x_2-x_1)^2+\cdots(x_N-x_{N-1})^2+(x_N-x_1)^2\right]+\frac{L}{2}\left[x_1^2+x_2^2+\cdots x_N^2\right]. \tag{2.4.1}$$

Problem 2.22: *Coupled blocks:* Two blocks of mass m and $m/2$ move along a frictionless track. The first (heavier) block is connected on the left to a stationary wall by a spring of Hookian constant K. It is connected on the right, via a similar spring, to the second (lighter) block, which has no further constraints.

1. Write the equations of motion for for the displacements $x_1(t)$ and $x_2(t)$ of the two blocks.
2. Find the frequencies of the normal modes of vibration of this system in terms of $\omega_0=\sqrt{K/m}$.
3. Find the amplitude ratios for each of the normal modes.

Problem 2.23: *Semi–open chain of blocks and springs:* Consider a chain of N blocks of mass m, connected by springs of Hookian constants K, which is *open at one end*, i.e. the last block is connected only on one side as in the following figure.

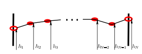


1. Write down the equations of motion for the displacements $\{u_\alpha\}$ of the N blocks (moving without friction), paying careful attention to the first and last blocks.
2. Show that a solution of the form $u_\alpha(k)=a(k)\sin(k\alpha)\cos[\omega(k)t]$ satisfies the equations of motion for $\alpha=1,2,\cdots,N-1$, and find the function $\omega(k)$.
3. Write down the equation that gives you the allowed (discrete) values of k, by requiring the above solution to also satisfy the equation of motion for u_N.

4. Find the N allowed values of k, and the corresponding normal mode frequencies.

   Hint: Think about the normal modes of a chain of $2N$ blocks which is fixed at both ends (as in lectures), and imagine the appropriate modes for each half.

Problem 2.24: *Open chain of beads on a string:* Consider a chain of N beads of mass m, glued to a string pulled with tension T, in which the two end beads are replaced by rings that can freely slide along two rods, as in the following figure.



1. Ignoring gravity and friction, write down the equations of motion for the vertical displacements $\{h_\alpha\}$ of the N beads, paying careful attention to the first and last beads.

2. Show that a solution of the form $h_\alpha(k) = a(k)\cos[k\alpha + \phi(k)]\cos[\omega(k)t]$ satisfies the equations of motion for $\alpha = 2, \cdots, N-1$, and find the function $\omega(k)$.

3. Write down the equations that give you the allowed (discrete) values of k, by requiring the above solution to also satisfy the equations of motion for h_1 and h_n.

4. Show that all n equations are simultaneously satisfied for $k = n\pi/N$ and $\phi(k) = -n\pi/(2N)$, i.e. for $h_\alpha(n) \propto \cos[n\pi(\alpha - 1/2)/N]$. What are the allowed values of n?