Chapter 2

Multiple variables

2.1 Two variables

Problem 2.1: Find the general solution to the system of differential equations

$$\frac{dx_1}{dt} = x_1 + x_2$$
$$\frac{dx_2}{dt} = 4x_1 + x_2,$$

that is, the homogeneous linear system of ODEs with constant coefficients

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

1. Compute the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$.

2. Use these eigenvalues and eigenvectors to form the change of basis matrix $\mathbf{O}$ and the diagonal matrix $\mathbf{D}$ such that
$$\mathbf{D} = \mathbf{O}^{-1}\mathbf{A}\mathbf{O}$$

3. Write down the general solution of the diagonalised system in which the transformed degrees of freedom $\mathbf{z} = \mathbf{C}^{-1}\mathbf{x}$ have decoupled
$$\frac{d\mathbf{z}}{dt} = \mathbf{D}\mathbf{z} \quad \Longrightarrow \quad \mathbf{z} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$
where the λ_i are the eigenvalues of $\mathbf{A}$.

4. Finally, construct the solution of the original (coupled) system as
$$\mathbf{x} = \mathbf{O}\mathbf{z}.$$

Problem 2.2: Solve the following system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Plot trajectories of the two solutions in the x_1–x_2 plane using the `ParametricPlot[]` function in mathematica.

Problem 2.3: Solve the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that this has only a single eigenvector so a second solution must be found.

Problem 2.4: The equation of motion for a system of a spring and mass with damping is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m, c, k are the mass, damping coefficient and spring force constant respectively, and all are positive constants.

1. Write this equation as a system of two first order equations for $u_1(t) = x(t)$ and $u_2(t) = \frac{du_1}{dt}$
2. Show that $u_1 = 0, u_2 = 0$ is a *critical point* where $\frac{du_1}{dt} = \frac{du_2}{dt} = 0$ and analyze the nature and stability of the critical point as a function of the parameters m, c and k.

2.2 Multiple variables

Problem 2.5: Show that the nth order differential equation

$$\frac{d^n x}{dt^n} = c_0 x + c_1 \frac{dx}{dt} + \cdots + c_{n-1} \frac{d^{n-1}x}{dt^{n-1}}, \quad c_j \in \mathbb{R}$$

can be written as a system of first order differential equation.

Problem 2.6: According to the Lorentz force law, the motion of a charge q in an electromagnetic field is given by

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where m denotes the mass and $\mathbf{v}$ the velocity (a 3-vector). Assume that

$$\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

are constant fields. Find the solution of the initial value problem $\mathbf{v}(t=0) = (0,0,0)$.

Problem 2.7: Express the dot-product and cross-product in terms of the summation convention. For the latter, introduce the Levi-Civita temsor ϵ_{ijk} for $\{i,j,k\} \in \{1,2,3\}$ which is totally antisymmetric under interchange of its indices ($\epsilon_{jik} = \epsilon ikj = -\epsilon_{kij} = -\epsilon_{ijk}$) and has $\epsilon 123 = 1$.

Problem 2.8: Show that

$$\begin{aligned} \epsilon^{jik}\epsilon_{lmn} &= \delta^j_l\delta^i_m\delta^k_n + \delta^j_m\delta^i_n\delta^k_l + \delta^j_n\delta^i_l\delta^k_m - \delta^j_l\delta^i_n\delta^k_m - \delta^j_n\delta^i_m\delta^k_l - \delta^j_m\delta^i_l\delta^k_n, \\ \epsilon^{ijk}\epsilon_{lmk} &= \delta^i_l\delta^j_m - \delta^i_m\delta^j_l, \\ \epsilon^{ijk}\epsilon_{ljk} &= 2\delta^i_l, \\ \epsilon^{ijk}\epsilon_{ijk} &= 6 \end{aligned} .$$

To do so, discuss the antisymmetry properties of the first equation and then use that equation to derive the rest.

Problem 2.9: Use index notation to prove the identities

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \\ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \end{aligned}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are 3-component vectors.

Problem 2.10: Consider the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$$

Find the eigenvalues and eigenvectors.

Problem 2.11: Let $x_1, x_2, x_3 \in \mathbb{R}$. Find the eigenvalues of the 2×2 matrix

$$\begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix}.$$

Problem 2.12: Let $\alpha, \beta, \gamma \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the 4×4 matrices

$$\begin{pmatrix} 0 & \cos(\alpha) & \cos(\beta) & \cos(\gamma) \\ \cos(\alpha) & 0 & 0 & 0 \\ \cos(\beta) & 0 & 0 & 0 \\ \cos(\gamma) & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \cosh(\alpha) & 0 & 0 & \sinh(\alpha) \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sinh(\alpha) & 0 & 0 & \cosh(\alpha) \end{pmatrix}.$$

Problem 2.13: Let $\alpha, \beta \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors of the 3×3 matrix

$$\begin{pmatrix} \alpha + \beta & 0 & \alpha \\ 0 & \alpha + \beta & 0 \\ \alpha & 0 & \alpha + \beta \end{pmatrix}$$

Problem 2.14: Let A be an $n \times n$ matrix over $\mathbb{C}$ with $A^2 = rA$, where $r \in \mathbb{C}$ and $r \neq 0$

1. Calculate e^{zA}, where $z \in \mathbb{C}$.
2. Let $U(z) = e^{zA}$. Let $z, z' \in \mathbb{C}$. Calculate $U(z)U(z')$

Problem 2.15: Let A be an $n \times n$ matrix over $\mathbb{C}$. We define $\sin(A)$ as

$$\sin(A) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} A^{2j+1}$$

If possible, find a 2×2 matrix B over the real numbers $\mathbb{R}$ such that

$$\sin(B) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}?$$

Problem 2.16: Let

$$A = \begin{pmatrix} 2 & 3 \\ 7 & -2 \end{pmatrix}$$

Calculate $\det e^A$

Problem 2.17: For every positive definite matrix A, there is a unique positive definite matrix Q such that $Q^2 = A$. The matrix Q is called the square root of A.

1. Find the square root of the matrix

$$B = \frac{1}{2} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

2. Find the square root of the positive definite 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Problem 2.18: Let A, B be $n \times n$ matrices and $t \in \mathbb{R}$. Show that

$$e^{t(A+B)} - e^{tA}e^{tB} = \frac{t^2}{2}(BA - AB) + \text{ higher order terms in } t.$$

2.3 Higher order coupled linear ODEs

Problem 2.19: Repeat the calculations of the normal modes of three and four block systems when the masses are not the same. Specifically, assume the ith block has mass $m_i = i$.

Problem 2.20: Repeat the calculations of the normal modes of three and four block systems when the masses are the same but the spring constants for each spring are different. Specifically, assume the spring connecting the ith block to the $(i+1)$th block has spring constant $K_i = 1/i$ for $i = 1, \ldots, n-1$ and that the spings connecting the first and last blocks to the rigid walls have spring constant $K = 1$.

2.4 Symmetries in matrices

Problem 2.21: *Periodic chain in a harmonic trap:* Find eigenvalues and eigenvectors of a periodic chain confined in a harmonic trap, i.e. subject to the potential

$$V\left(x_1, \cdots x_N\right) = \frac{K}{2}\left[(x_2 - x_1)^2 + \cdots (x_N - x_{N-1})^2 + (x_N - x_1)^2\right] + \frac{L}{2}\left[x_1^2 + x_2^2 + \cdots x_N^2\right] . \tag{2.4.1}$$

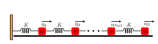
Problem 2.22: *Coupled blocks:* Two blocks of mass m and $m/2$ move along a frictionless track. The first (heavier) block is connected on the left to a stationary wall by a spring of Hookian constant K. It is connected on the right, via a similar spring, to the second (lighter) block, which has no further constraints.

1. Write the equations of motion for for the displacements $x_1(t)$ and $x_2(t)$ of the two blocks.

2. Find the frequencies of the normal modes of vibration of this system in terms of $\omega_0 = \sqrt{K/m}$.

3. Find the amplitude ratios for each of the normal modes.

Problem 2.23: *Semi–open chain of blocks and springs:* Consider a chain of N blocks of mass m, connected by springs of Hookian constants K, which is *open at one end*, i.e. the last block is connected only on one side as in the following figure.



1. Write down the equations of motion for the displacements $\{u_\alpha\}$ of the N blocks (moving without friction), paying careful attention to the first and last blocks.

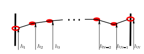
2. Show that a solution of the form $u_\alpha(k) = a(k)\sin(k\alpha)\cos[\omega(k)t]$ satisfies the equations of motion for $\alpha = 1, 2, \cdots, N-1$, and find the function $\omega(k)$.

3. Write down the equation that gives you the allowed (discrete) values of k, by requiring the above solution to also satisfy the equation of motion for u_N.

4. Find the N allowed values of k, and the corresponding normal mode frequencies.

   Hint: Think about the normal modes of a chain of $2N$ blocks which is fixed at both ends (as in lectures), and imagine the appropriate modes for each half.

Problem 2.24: *Open chain of beads on a string:* Consider a chain of N beads of mass m, glued to a string pulled with tension T, in which the two end beads are replaced by rings that can freely slide along two rods, as in the following figure.



1. Ignoring gravity and friction, write down the equations of motion for the vertical displacements $\{h_\alpha\}$ of the N beads, paying careful attention to the first and last beads.

2. Show that a solution of the form $h_\alpha(k) = a(k)\cos[k\alpha + \phi(k)]\cos[\omega(k)t]$ satisfies the equations of motion for $\alpha = 2, \cdots, N-1$, and find the function $\omega(k)$.

3. Write down the equations that give you the allowed (discrete) values of k, by requiring the above solution to also satisfy the equations of motion for h_1 and h_n.

4. Show that all n equations are simultaneously satisfied for $k = n\pi/N$ and $\phi(k) = -n\pi/(2N)$, i.e. for $h_\alpha(n) \propto \cos[n\pi(\alpha - 1/2)/N]$. What are the allowed values of n?