3.2 Solving PDEs

Problem 3.3: Show using a change of variables to $\xi = x + ct$ and $\eta = x - ct$ and subsequent direct integration that

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

has the general solution

$$u(x,t) = f(x - ct) + g(x + ct)$$

where f and g are twice differentiable functions.

Problem 3.4: Consider a rod of length L. Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for} \quad 0 < t$$

and the initial conditions

$$u(x,0) = u_0(x) \quad \text{for} \quad 0 < x < L$$

where u_0 is some known function.

Problem 3.5: The wave equation descriibing a plucked string in the presence of damping takes the form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -\kappa \frac{\partial u}{\partial t} - \text{ for } 0 < x < L$$
$$u(0,t) = 0 \text{ and } u(L,t) = 0$$

where the drag coefficient κ is a non-negative constant.

Show through separation of variables that for small κ, the solutions are of the form

$$u_n(x,t) = e^{-\kappa t/2} \left[A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)\right] \sin\left(\frac{n\pi}{L} x\right)$$

wher the γ_n are constants that you should find the form of. Similarly, show for large κ the solutions are of the form

$$u_n(x,t) = C_n e^{-\kappa t/2} \sin\left(\frac{n\pi}{L} x\right).$$

and explain what defines large and small.