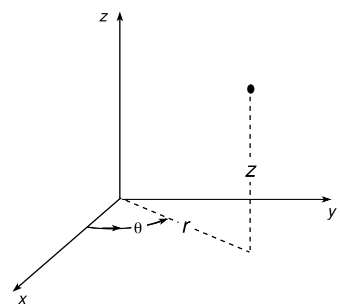
3.5 Coordinate transformations

Problem 3.17: *Polar Coordinates:* Make sure you are familiar with the following expressions for the gradient, divergence, and curl in polar coordinates (r, θ, z):

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial z}\right), \quad \text{for a scalar field } \Phi(r,\theta,z),$$

$$\nabla\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}, \quad \text{for a vector field } \vec{v} = (v_r, v_\theta, v_z),$$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}, \quad \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right).$$



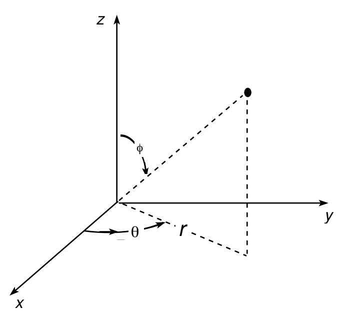
1. Write down the expression for $\nabla^2\Phi(r,\theta,z)$ in polar coordinates.
2. Verify by explicit calculation in polar coordinates that
$$\nabla\cdot(\nabla\times\vec{v}) = 0.$$
3. Verify by explicit calculation of both sides in polar coordinates that
$$\nabla\times(\nabla\times\vec{v}) = \nabla(\nabla\cdot\vec{v}) - \nabla^2\vec{v}.$$
4. Calculate the gradient of $\Phi = \sqrt{r^2+z^2}$, the divergence of $\vec{v} = (0, 0, 1/\sqrt{r^2+z^2})$, and the curl of $\vec{u} = (0, 2\pi/r, 0)$.

Problem 3.18: *Spherical Coordinates:* Make sure you are familiar with the following expressions for the gradient, divergence, and curl in spherical coordinates (r, θ, ϕ):

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right), \quad \text{for a scalar field } \Phi(r, \theta, z),$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}, \quad \text{for a vector field } \vec{v} = \left(v_r, v_\theta, v_\phi\right),$$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{\cot\theta}{r}v_\phi - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}, \quad \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r} - \frac{\partial v_\phi}{\partial r}, \quad \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right).$$



1. Write down the expression for $\nabla^2\Phi(r, \theta, \phi)$ in spherical coordinates.
2. Verify by explicit calculation in spherical coordinates that

$$\nabla\cdot(\nabla\times\vec{v}) = 0.$$

3. Verify by explicit calculation of both sides in spherical coordinates that

$$\nabla\times(\nabla\times\vec{v}) = \nabla\left(\nabla\cdot\vec{v}\right) - \nabla^2\vec{v}.$$

4. Calculate the divergence of $\vec{v} = (\cos\theta/r, \sin\theta/r, 0)$, and the Laplacian of $\sin(kr)/r$.