Chapter 3

Continuous fields

3.1 From particles to fields

Problem 3.1: For the functional

$$V[f] = \int dx \left[af(x)^2 + bf(x)f'(x) + cf''(x)^2 f'(x)\right]$$

calculate the functional derivatives $\frac{\delta V}{\delta f}$ and $\frac{\delta^2 V}{\delta f^2} = \frac{\delta}{\delta f}\frac{\delta V}{\delta f}$ assuming a, b, and c are constants.

Problem 3.2: Perform the generalised gradient expansion (3.1.10) to determine the force density acting

3.2 Solving PDEs

Problem 3.3: Show using a change of variables to $\xi = x + ct$ and $\eta = x - ct$ and subsequent direct integration that

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

has the general solution

$$u(x,t) = f(x - ct) + g(x + ct)$$

where f and g are twice differentiable functions.

Problem 3.4: Consider a rod of length L. Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for} \quad 0 < t$$

and the initial conditions

$$u(x,0) = u_0(x) \quad \text{for} \quad 0 < x < L$$

where u_0 is some known function.

Problem 3.5: The wave equation descriibing a plucked string in the presence of damping takes the form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -\kappa \frac{\partial u}{\partial t} - \text{ for } 0 < x < L$$
$$u(0,t) = 0 \text{ and } u(L,t) = 0$$

where the drag coefficient κ is a non-negative constant.

Show through separation of variables that for small κ, the solutions are of the form

$$u_n(x,t) = e^{-\kappa t/2} \left[A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)\right] \sin\left(\frac{n\pi}{L} x\right)$$

wher the γ_n are constants that you should find the form of. Similarly, show for large κ the solutions are of the form

$$u_n(x,t) = C_n e^{-\kappa t/2} \sin\left(\frac{n\pi}{L} x\right).$$

and explain what defines large and small.

3.3 Fourier analysis

Problem 3.6: Show the following relations

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0$$

$$\int_{-\pi}^{\pi} \cos nt \cos mt dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \\ 2\pi & n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \begin{cases} 0 & n \neq m, n = m = 0 \\ \pi & n = m \end{cases}$$

Problem 3.7: Find the Fourier series for the following examples

1. $f(x) = |x|, -\pi < x < \pi$
2. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$
3. $f(x) = 1 + x$ on $[-\pi, \pi]$
4. $f(x) = 1 + \sin^2 t$
5. $f(x) = \begin{cases} 1 & -1 \leq x < 0 \\ \frac{1}{2} & x = 0 \quad \text{on } [-1, 1] \\ x & 0 < x \leq 1 \end{cases}$
6. $f(x) = \begin{cases} -1 & -3 \leq x < 0 \\ 1 & 0 < x \leq 3 \end{cases}$ on [-3,3]
7. $f(x) = x^2$ on $[-\pi, \pi]$
8. $f(x) = f(x + 2), f(x) = (x - 1)(x - 3)$ on [1,3] .
9. $f(x) = x$ on [0,1] .
10. $f(t) = \begin{cases} \frac{4}{\pi} t & 0 \leq t < \frac{\pi}{2} \\ \frac{-4}{\pi} t & \frac{-\pi}{2} \leq t \leq 0 \end{cases}$

3.4 Scalar fields in higher dimensions

Problem 3.8: Find the temperature distribution, $u = u(\vec{x}, t)$, on a disk of radius R when the initial temperature distribution is known and the temperature on the boundary of the disk is kept at zero degrees. Working in polar coordinates appropriate for the symmetry, this means finding $u = u(r, \theta, t)$ satisfying the partial differential equation

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \quad \text{for} \quad 0 \leq r < R \quad \text{and} \quad t > 0$$

(where κ is some positive constant depending on the material making up the disk), along with the boundary conditions

$$u(R, \theta, t) = 0 \quad \text{for} \quad t > 0$$

as well as the initial condition

$$u(r, \theta, 0) = u_0(r, \theta)$$

for $0 \leq r < a$ and $t > 0$ where u_0 is some known function.

Problem 3.9: *Energy in Waves:* Consider longitudinal deformations $u(x, t)$, of a spring. There is a local *kinetic energy density* $\rho \dot{u}^2/2$, where ρ is the mass per unit length, and a local *potential energy density* $Ku'^2/2$, where K is the local analog of the Hookian constant. The net energy of a spring of length L can thus be written as

$$E = \int_0^L dx \left[\frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{k}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right].$$

1. Assume that the ends of the spring are fixed , such that $u(x = 0) = u(x = L) = 0$ at all times. Write down the deformation $u^{(n)}(x, t)$ for the nth normal mode of vibration.
2. Calculate the net kinetic energy, and the net potential energy for a normal mode, and show that they have the same time average.

Problem 3.10: *Plucked String:* In a guitar or harpsichord,[1] tones are produced by *plucking* a string. Consider a string of length L, fixed at both ends, which is plucked at a distance d from one end by an amount w, and then released, so that the initial conditions are given by

$$h(x, t = 0) = \begin{cases} \dfrac{wx}{d} & \text{for} \quad 0 \leq x \leq d \\ \dfrac{w(L - x)}{(L - d)} & \text{for} \quad d \leq x \leq L \end{cases} , \qquad \text{and} \qquad \dot{h}(x, t = 0) = 0\,.$$

[1] A stringed instrument resembling a grand piano but with two keyboards and two or more strings for each note and producing tones by the plucking of strings with plectra.

1. Show that the initial profile can be decomposed in a Fourier series as

$$h(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \qquad \text{with} \qquad A_n = \frac{2wL^2}{n^2\pi^2 d(L-d)} \sin\left(\frac{n\pi d}{L}\right).$$

2. Assuming a wave velocity of v, write down the expression for $h(x,t)$ at all times.

Problem 3.11: *Hammered String:* In a piano, tones are generated by a hammer that strikes a string of length L at a distance d from one end (typically $d = L/7$). A simplified model of this is to assume an initial condition in which the string is straight, $h(x, t = 0) = 0$, but acquires an initial velocity at the point of impact by the hammer, approximated by[2]

$$\dot{h}(x, t = 0) = u_0 \delta(x - d).$$

1. Assuming a wave velocity of v for the piano string, write down the expression for the string shape $h(x,t)$ at all times, as a Fourier series.
2. Show that, quite generally, the kinetic energy of the string can be written as a sum of contributions from individual normal modes (without any cross terms between modes).
3. Give the expression for the time averaged energy in the modes of the hammered string.

Problem 3.12: *Wave transients:* A string of length L is initially at rest, with one end fixed to a wall. At time $t = 0$, the other end of the string is forced to oscillate as $h(x = 0, t) = w \sin(\omega t)$, where ω is not a resonance frequency.

1. Write down the initial conditions, as well as the boundary conditions, for the shape of the string, $h(x,t)$.
2. Show that the wave equation and the boundary conditions, are satisfied by

$$h(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi v}{L} t + \theta_n\right) + h_s(x,t),$$

where $h_s(x,t)$ is the solution obtained without considering the initial conditions.

[2]The so-called δ-function is the limit of functions which are peaked around a point, and zero everywhere else, such that for any function $f(x)$,

$$\int dx f(x)\delta(x-d) = f(d).$$

3. Find the parameters $\{A_n\}$ and $\{\theta_n\}$ by requiring that the initial conditions are satisfied.

Problem 3.13: *Travelling waves:* Consider a very long wire of mass ρ per unit length, under a strong tension T. The initial shape and velocity of the wire are given by

$$h(x,0) = w \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \text{and} \qquad \dot{h}(x,0) = \frac{wv}{\sigma^2}\, x \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where $v = \sqrt{T/\rho}$ is the wave velocity.

1. Find the shape $h(x,t)$, and its velocity $\dot{h}(x,t)$, at all times.
2. Calculate the net kinetic energy of the wire at all times.[3]
3. Calculate the net potential energy of the wire at all times.
4. What is the shape of the wire at time t, if the initial conditions are changed to

$$h(x,0) = w \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \text{and} \qquad \dot{h}(x,0) = 0.$$

Problem 3.14: *Confined Cable:* A massive cable is under a tension T and rests along the bottom of a rigid semi-cylindrical groove. The action of gravity provides an additional restoring force, such that the equation of motion for the displacements $h(x,t)$ of the cable is given by

$$\mu \frac{\partial^2 h}{\partial t^2} = -\frac{\mu g}{R} h + T \frac{\partial^2 h}{\partial x^2},$$

where μ is the mass per unit length of the cable, g is the acceleration due to gravity, and R is the radius of the confining groove.

1. Show that a separable solution of the form $h(x,t) \propto \cos(\omega t + \phi)\sin(kx + \theta)$ satisfies the above equation, and give the dispersion relation $\omega(k)$.
2. Note that there are no propagating waves for $\omega < \omega_0 = \sqrt{g/R}$. It is still possible, however, to find separable solutions of the form $h(x,t) \propto \cos(\omega t + \phi)X(x)$. What are possible solutions for $X(x)$ in this case?

[3] You may need the following integrals:

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}, \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx x^2 e^{-x^2} = \sqrt{\pi}/2.$$

3. One end of the cable is forced to moves as $h(x=0,t) = a\cos(\omega t)$, at a frequency $\omega < \omega_0$. Find the steady-state solution $h(x,t)$, which goes to zero at large x.

4. For the above steady-state solution calculate the velocity of the string $\dot{h}(0,t)$, and the transverse force $F_h(0,t)$, at $x=0$. Hence show that the time average power input $\overline{P}_{in} = \overline{\dot{h}(0,t)F_h(0,t)}$ is zero.

Problem 3.15: *Viscous boundary:* A long string is attached at $x=0$ by a massless ring to a wire covered with grease, and is subject to a tension T. The ring experiences a drag force $F_h = -b\dot{h}(x=0,t)$, when the end of the string moves.

1. By equating the forces acting on the (massless) ring, find the boundary condition at $x=0$.

2. Show that the boundary condition is satisfied by a distortion which is the sum of an incident pulse $f(x+vt)$ (arriving from $+x$ direction), and a reflected pulse $g(x-vt)$, where v is the wave velocity. Find g in terms of f.

3. Explain the limiting behaviors for $b \to 0$, and $b \to \infty$.

4. For what value of b is there no reflection?

Problem 3.16: *Generating waves:* A very long *inextensible* string is held horizontally at a tension T by one end passing over a pulley and being attached to a mass $M = T/g$. The other end is fixed to a support which at $t=0$ starts to move as $h(x=0,t>0) = H\sin(\omega t)$.

1. Assuming a wave velocity v, write down the expression for the travelling wave along the string, for times before it reaches the other end.

2. Show that the inextensible string is pulled in by an amount

$$s = \int_0^{vt} dx \left[\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2} - 1\right] \approx \frac{1}{2}\int_0^{vt} dx \left(\frac{\partial h}{\partial x}\right)^2,$$

and hence calculate the average vertical velocity of the mass.

3. Calculate the velocity of the string $\dot{h}$, and the transverse force $F_h(0,t)$ at $x=0$, and hence obtain the time average power input $\overline{P}_{in} = \overline{\dot{h}(0,t)F_h(0,t)}$. Is this equal to the rise in the potential energy of the mass? If not, where is the remaining energy?

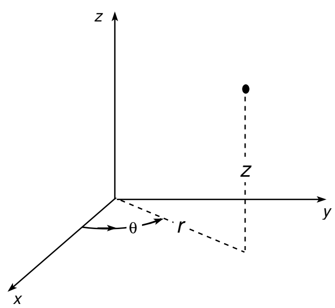
3.5 Coordinate transformations

Problem 3.17: *Polar Coordinates:* Make sure you are familiar with the following expressions for the gradient, divergence, and curl in polar coordinates (r, θ, z):

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial z}\right), \quad \text{for a scalar field } \Phi(r, \theta, z),$$

$$\nabla \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}, \quad \text{for a vector field } \vec{v} = (v_r, v_\theta, v_z),$$

$$\nabla \times \vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}, \quad \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right).$$



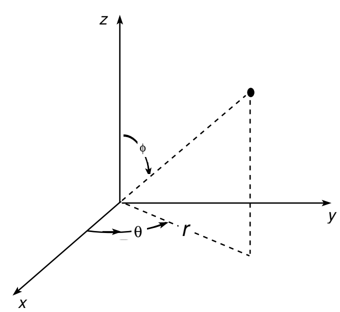
1. Write down the expression for $\nabla^2\Phi(r, \theta, z)$ in polar coordinates.
2. Verify by explicit calculation in polar coordinates that
$$\nabla \cdot (\nabla \times \vec{v}) = 0.$$
3. Verify by explicit calculation of both sides in polar coordinates that
$$\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2\vec{v}.$$
4. Calculate the gradient of $\Phi = \sqrt{r^2 + z^2}$, the divergence of $\vec{v} = (0, 0, 1/\sqrt{r^2 + z^2})$, and the curl of $\vec{u} = (0, 2\pi/r, 0)$.

Problem 3.18: *Spherical Coordinates:* Make sure you are familiar with the following expressions for the gradient, divergence, and curl in spherical coordinates (r, θ, ϕ):

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right), \quad \text{for a scalar field } \Phi(r, \theta, z),$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}, \quad \text{for a vector field } \vec{v} = \left(v_r, v_\theta, v_\phi\right),$$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{\cot\theta}{r}v_\phi - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}, \quad \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r} - \frac{\partial v_\phi}{\partial r}, \quad \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right).$$



1. Write down the expression for $\nabla^2\Phi(r, \theta, \phi)$ in spherical coordinates.
2. Verify by explicit calculation in spherical coordinates that
$$\nabla\cdot(\nabla\times\vec{v}) = 0.$$
3. Verify by explicit calculation of both sides in spherical coordinates that
$$\nabla\times(\nabla\times\vec{v}) = \nabla\left(\nabla\cdot\vec{v}\right) - \nabla^2\vec{v}.$$
4. Calculate the divergence of $\vec{v} = (\cos\theta/r, \sin\theta/r, 0)$, and the Laplacian of $\sin(kr)/r$.

3.6 Dynamics of vector fields

Problem 3.19: *Maxwell's Equations:* Maxwell's equations relate the electric and magnetic fields $\vec{E}$ and $\vec{B}$, to the charge and current densities ρ and $\vec{j}$, by the differential equations

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j} \\ \nabla \cdot \vec{B} = 0 \end{cases} .$$

1. Show that these equations imply the following relation between ρ and $\vec{j}$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0,$$

which represents the conservation of charge.

2. Write down explicitly the 8 equations for $\vec{E} = (E_r, E_\theta, E_z)$ and $\vec{B} = (B_r, B_\theta, B_z)$ in cylindrical coordinates (r, θ, z).

3. Write down explicitly the 8 equations for $\vec{E} = (E_r, E_\theta, E_\phi)$ and $\vec{B} = (B_r, B_\theta, B_\phi)$ in spherical coordinates (r, θ, ϕ).