4.3 Many random variables

1. *Optimal selection:* In many specialized populations, there is little variability among the members. Is this a natural consequence of optimal selection?

   (a) Let $\{r_\alpha\}$ be n random numbers, each independently chosen from a probability density $p(r)$, with $r \in [0, 1]$. Calculate the probability density $p_n(x)$ for the largest value of this set, i.e. for $x = \max\{r_1, \cdots, r_n\}$.

   (b) If each r_α is uniformly distributed between 0 and 1, calculate the mean and variance of x as a function of n, and comment on their behavior at large n.

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2. *Benford's law* describes the observed probabilities of the *first digit* in a great variety of data sets, such as stock prices. Rather counter-intuitively, the digits 1 through 9 occur with probabilities 0.301, .176, .125, .097, .079, .067, .058, .051, .046 respectively. The key observation is that this distribution is invariant under a change of scale, e.g. if the stock prices were converted from dollars to persian rials! Find a formula that fits the above probabilities on the basis of this observation.

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3. *Directed random walk:* The motion of a particle in three dimensions is a series of independent steps of length ℓ. Each step makes an angle θ with the z axis, with a probability density $p(\theta) = 2\cos^2(\theta/2)/\pi$; while the angle ϕ is uniformly distributed between 0 and 2π. (Note that the solid angle factor of $\sin\theta$ is already included in the definition of $p(\theta)$, which is correctly normalized to unity.) The particle (walker) starts at the origin and makes a large number of steps N.

   (a) Calculate the expectation values $\langle z \rangle$, $\langle x \rangle$, $\langle y \rangle$, $\langle z^2 \rangle$, $\langle x^2 \rangle$, and $\langle y^2 \rangle$, and the covariances $\langle xy \rangle$, $\langle xz \rangle$, and $\langle yz \rangle$.

   (b) Use the central limit theorem to estimate the probability density $p(x, y, z)$ for the particle to end up at the point (x, y, z).

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