1.1 Taylor Expansions

1.1.1 Variable $\Longrightarrow$ Function

The position of a moving particle is an example of a variable that proceeds *continuously* from one point in space to another, from one moment in time to the next. Mathematically, functions that describe such quantities are *analytic*, and can be expanded as a *Taylor series*. For example, the function $x(t)$ quantifying variations in location of a particle in time can be written as

$$x(t) = x_0 + x_1 t + \frac{x_2}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_n}{n!}t^n. \tag{1.1.1}$$

The set of coefficients $\{x_n\}$ in the expansion can be obtained by taking successive derivatives of the function. Recalling that $\frac{dt^p}{dt} = pt^{p-1}$, we obtain

$$\frac{dx(t)}{dt} = x_1 + x_2 t + \frac{x_3}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_{n+1}}{n!}t^n, \quad \Longrightarrow \quad \left.\frac{dx(t)}{dt}\right|_{t=0} = x_1. \tag{1.1.2}$$

Taking more derivatives removes further terms from the start of the series, such that

$$\frac{d^p x(t)}{dt^p} = x_p + x_{p+1} t + \frac{x_{p+2}}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_{n+p}}{n!}t^n, \tag{1.1.3}$$

leading to

$$x_n = \left.\frac{d^n x}{dt^n}\right|_{t=0}. \tag{1.1.4}$$

The above represents that Taylor expansion around $t = 0$. Naturally, we can also expand around any other point $t = t_0$, as

$$x(t) = \tilde{x}_0 + \tilde{x}_1(t - t_0) + \frac{\tilde{x}_2}{2!}(t - t_0)^2 + \frac{\tilde{x}_3}{3!}(t - t_0)^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{\tilde{x}_n}{n!}(t - t_0)^n, \tag{1.1.5}$$

with

$$\tilde{x}_n = \left.\frac{d^n x}{dt^n}\right|_{t=t_0}. \tag{1.1.6}$$

For the time being we shall not deal with *non-analytic* functions where such an expansion is not possible.