1.1.2 Small displacements $\Longrightarrow$ Linear response

Surprisingly, we can usually get quite far by approximating a Taylor expansion with its first terms. Of course, the trick is to do this for the right quantity in the appropriate limit, as we shall see shortly. An important example of this in mechanics is the famous *Hooke's law* which states that the force F is proportional (and opposed) to the displacement. Imagine pushing a spring, or any other elastic body, by a displacement x away from its equilibrium state. The spring responds by exerting a force $F(x)$, which can be represented by the Taylor series

$$F(x) = f_0 + f_1 x + \frac{f_2}{2!}x^2 + \cdots \approx -Kx, \tag{1.1.7}$$

where we have used common sense to decide that the first coefficient f_0 is zero (at equilibrium), and the second is negative ($K = -f_1 > 0$). Naturally higher order terms are present in any material, and will invalidate the Hookian approximation when the displacement exceeds say $x^* \approx f_1/f_2$. As long as the contribution from higher order terms in the series is small, which will always be the case for small enough displacements, we can use this approximation. A *linear response* to perturbations is quite frequently used in physics as it is amenable to analytic computations that usually provide much insight. It is, however, important to be aware of the limits to validity of linearized models; the non-linear regime is harder to handle, and could lead to very different behaviors (e.g when a spring breaks).

In mechanical systems, we can relate the force to the derivative of another function, the potential energy $V(x)$ by

$$F(x) = -\frac{dV(x)}{dx}. \tag{1.1.8}$$

Note that if we construct a Taylor series for the potential energy corresponding to displacements *around an equilibrium* point, we get

$$V(x) = V_0 + \frac{K}{2}x^2 + \text{higher order terms}\,. \tag{1.1.9}$$

Quite generally, expansions around an equilibrium position, corresponding to zero force, start with a quadratic term. Ignoring higher order terms leads to a quadratic or *harmonic* potential. The ideal Hookian spring thus has a linear force law, and a harmonic potential energy.