1.1.3 Instantaneous reaction $\Longrightarrow$ Simple differential equations

We noted that displacing a particle from its equilibrium position leads to a response, quantified by the force $F(x)$. The force in turn causes variations of the coordinate in time as $x(t)$, captured by the Taylor series introduced at the beginning. The leading terms in this series are the velocity $x_1 = v = dx/dt \equiv \dot{x}$ and the acceleration $x_2 = a = d^2x/dt^2 \equiv \ddot{x}$. Assuming that the evolution of the coordinate x at each time is constrained by a relation between these time derivatives and the instananeous force, naturally leads to *differential equations*. While the laws of physics encode these variations through Newton's equation of motion, it is instructive to follow an agnostic perspective.

(1) The simplest assumption we can make is that the velocity of the particle at each time is proportional to the force at that time, i.e. $F(\text{at time } t) = F(x \text{ at time } t) = F(x(t))$, leading to the ordinary differential equation (ODE)

$$\dot{x} = \mu F(x). \tag{1.1.10}$$

This is actually a very good description for a particle moving in a viscous fluid like oil, and μ in the above equation is known as the *mobility*. Let us further assume that at $t = 0$, the particle is displaced by a small amount to x_0. To find its position $x(t)$ as a function of time, we need to solve the *first order differential equation*[1]

$$\dot{x} \approx -\mu K x \equiv -\gamma x. \tag{1.1.11}$$

The force has been linearized, and we have introduced the parameter γ with dimensions of inverse time.

[1]The order of a differential equation is defined by the highest derivative term appearing in the equation, e.g. if $d^n x/dt^n$ is the highest derivative, the ODE is of n^{th} order.