1.1.4 The exponential solution

To find the particle position $x(t)$ we now need to solve the *linear differential equation* $\dot{x} = -\gamma x$. One way of doing this is to develop a Taylor series for $x(t)$ around $t = 0$, whose coefficients are derivatives evaluated at $t = 0$. The differential equation allows us to calculate these derivatives easily, giving in particular

$$\frac{dx}{dt}(t = 0) = -\gamma x(0) = -\gamma x_0. \tag{1.1.12}$$

Higher derivatives can be successively related to lower derivatives by taking derivatives of the differential equation, as

$$\frac{d^2x}{dt^2}(0) = -\gamma \frac{dx}{dt}(0) = +\gamma^2 x_0, \tag{1.1.13}$$

and the general term is

$$\frac{d^n x}{dt^n}(0) = -\gamma \frac{d^{n-1}x}{dt^{n-1}}(0) = (-\gamma)^n x_0. \tag{1.1.14}$$

The solution can thus be obtained from the series

$$x(t) = x_0 - \gamma x_0 t + \frac{\gamma^2}{2!} x_0 t^2 + \cdots = x_0 \sum_{n=0}^{\infty} \frac{(-\gamma t)^n}{n!} = x_0 e^{-\gamma t}. \tag{1.1.15}$$

For the final step, we have employed the series expansion for the exponential function.

• *The exponential function has the nice property of keeping its form under differentiation. It thus appears quite generally as a solution to all linear differential equations.*