1.1.5 Time reversal symmetry

The above solution describes a displacement that decays to zero over a characteristic time $\tau = 1/\gamma$. While this may be an acceptable solution to a creature living in viscous oil, a being in free space notices that things typically do not come to rest, but will instead oscillate around their equilibrium position for long times. We then make the crucial observation that an oscillation looks the same going forward or backward in time. If the laws of nature have such *time reversal symmetry* then we should use an equation of motion that respects this, and does not change under $t \to -t$. Since the velocity changes sign under this transformation, the earlier proposed Eq. (1.1.10) does not describe such a situation. Insisting on time reversal symmetry as a property of nature then leads to Newton's law of motion in which the acceleration, $a = d^2x/dt^2 \equiv \ddot{x}$ (which is invariant under $t \to -t$) is proportional to force, i.e.

$$m\ddot{x} = F(x)\,, \tag{1.1.16}$$

where m is the mass. In the linear regime $F(x) \approx -Kx$, with K as the Hookian coefficient, we thus arrive to the *second order differential equation*

$$\ddot{x} = -\omega_0^2 x\,, \tag{1.1.17}$$

where $\omega_0 = \sqrt{K/m}$ has dimensions of inverse time. This equation describes *Simple Harmonic Oscillations* (SHOs) as shown next.

We can obtain the solution to this equation by the same series method as before. However, while even derivatives at $t = 0$ can be obtained from the initial displacement x_0, the odd derivatives are related to the initial velocity v_0. The complete solution is thus

$$\begin{aligned} x(t) &= x_0 + v_0 t - \frac{\omega_0^2}{2!}x_0 t^2 - \frac{\omega_0^2}{3!}v_0 t^3 + \cdots \\ &= x_0\left[1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} + \cdots\right] + \frac{v_0}{\omega_0}\left[(\omega_0 t) - \frac{(\omega_0 t)^3}{3!} + \cdots\right] \\ &= x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\,. \end{aligned} \tag{1.1.18}$$

For the final step we have assumed familiarity with the series expansions for sine and cosine functions. The solution in this case is a periodic function, i.e. it represents oscillations around the origin. The function repeats with a *period* of

$$T = \frac{2\pi}{\omega_0} = \nu^{-1}, \tag{1.1.19}$$

where the *frequency* ν has units of Hertz (inverse second). The parameter ω_0 is the *angular frequency* and has units of radians per second.

• *Note that a solution to the second order differential equation depends on two initial conditions, x_0 and v_0. From the series approach you can see that for n^{th} order differential equations, one needs to specify n initial conditions.*

⋆ *Pendulum:* A mass m at the end of a string of length ℓ is disturbed from equilibrium. Let us denote θ the angle from the vertical. The tangential acceleration is $\ell\ddot{\theta}$, while the tangential component of the restoring force due to gravity is $mg\sin\theta$. From Newton's law we have

$$m\ell\ddot{\theta} = -mg\sin\theta \approx -mg\theta, \tag{1.1.20}$$

where the last step comes from linearizing the force. This can be cast in the standard form $\ddot{\theta} = -\omega_0^2\theta$ for SHO, with angular frequency $\omega_0 = \sqrt{g/\ell}$. Note that the angular frequency is independent of mass (or shape) of the object. This is a physical principle that could not be guessed from mathematics alone, and thus points to existence of yet another symmetry of nature.

(3) A more general equation describing the motion of particle in a fluid includes both inertial and friction terms, taking the form

$$F(x) = m\ddot{x} + \frac{1}{\mu}\dot{x} + \cdots . \tag{1.1.21}$$

While the Newtonian term, $m\ddot{x}$ is time reversal symmetric, inclusion of friction via $\dot{x}/\mu$ removes this symmetry. A linear response around equilibrium then leads to damped oscillations that will be quantified later. Note, however, that in the spirit of a series expansion we can in principle add to the right hand side higher order derivatives, such as d^3x/dt^3, as well as nonlinear terms such as $\ddot{x}\dot{x}^2$. Indeed the former appears as a form of quantum friction in vacuum, while near relativistic speeds allow for the latter; the latter being negligible at ordinary settings dealing with velocities much smaller than the speed of light. The lesson is that even commonly occurring equations of motion may be regarded as expressions of low order terms in a series expansion, and thus as mathematical constructs that transcend particular physics applications.