1.1.6 The Energy Function

For future reference we make the following observations:

(1) Any first order differential equation of the form $\dot{x} = \mu F(x)$ can be cast as

$$\dot{x} = -\mu \frac{dV(x)}{dx}, \quad \text{with} \quad V(x) = -\int^{x} dx' F(x'). \tag{1.1.22}$$

As its argument changes with time, so does the potential $V(x(t))$, and its variations are obtained using the *chain rule* of differentiation as

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = -\mu \left(\frac{dV}{dx} \right)^2 \leq 0. \tag{1.1.23}$$

For $\mu > 0$, the value of potential can only decrease with time, and Eq. (1.1.22) describes *gradient descent* in the potential $V(x)$. The coordinate x proceeds towards a stationary (equilibrium) state corresponding to closest minimum of the potential $V(x)$. (The stationary point at a local maximum of $V(x)$ is referred to as an unstable equilibrium point.)

(2) For any second order differential equation of the form $m\ddot{x} = F(x) = -dV(x)/dx$, we can define a first integral by multiplying both sides of the equation with $\dot{x}$, and rearranging as

$$0 = m\dot{x}\ddot{x} + \dot{x}\frac{dV(x)}{dx} = \frac{d}{dt}\left[m\frac{\dot{x}^2}{2} + V(x) \right] \equiv \frac{dE}{dt}\,. \tag{1.1.24}$$

This immediately implies that the quantity

$$E(t) = m\frac{\dot{x}^2}{2} + V(x) = E_0, \tag{1.1.25}$$

is a constant of motion that does not change over time. In the context of a particle, E corresponds to the sum of a kinetic energy $m\dot{x}^2/2$, and a potential energy $V(x)$. For small distortions around an equilibrium position ($x = \dot{x} = 0$), the energy can then be expanded as

$$E(t) = \frac{m}{2}\dot{x}^2 + \frac{K}{2}x^2 + \text{higher order terms} = E_0\,. \tag{1.1.26}$$

Conservation of energy then leads to

$$0 = \frac{dE}{dt} \approx M\dot{x}\ddot{x} + Kx\dot{x} = \dot{x}\left(M\ddot{x} + Kx\right). \tag{1.1.27}$$

Setting the term in the brackets to zero reproduces the equation of motion, in this case again describing SHOs with $\omega_0 = \sqrt{K/M}$.