Chapter 1

One variable

1.1 Taylor Expansions

1.1.1 Variable $\Longrightarrow$ Function

The position of a moving particle is an example of a variable that proceeds *continuously* from one point in space to another, from one moment in time to the next. Mathematically, functions that describe such quantities are *analytic*, and can be expanded as a *Taylor series*. For example, the function $x(t)$ quantifying variations in location of a particle in time can be written as

$$x(t) = x_0 + x_1 t + \frac{x_2}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_n}{n!}t^n. \tag{1.1.1}$$

The set of coefficients $\{x_n\}$ in the expansion can be obtained by taking successive derivatives of the function. Recalling that $\frac{dt^p}{dt} = pt^{p-1}$, we obtain

$$\frac{dx(t)}{dt} = x_1 + x_2 t + \frac{x_3}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_{n+1}}{n!}t^n, \quad \Longrightarrow \quad \left.\frac{dx(t)}{dt}\right|_{t=0} = x_1. \tag{1.1.2}$$

Taking more derivatives removes further terms from the start of the series, such that

$$\frac{d^p x(t)}{dt^p} = x_p + x_{p+1}t + \frac{x_{p+2}}{2!}t^2 + \frac{x_3}{3!}t^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{x_{n+p}}{n!}t^n, \tag{1.1.3}$$

leading to

$$x_n = \left.\frac{d^n x}{dt^n}\right|_{t=0}. \tag{1.1.4}$$

The above represents that Taylor expansion around $t = 0$. Naturally, we can also expand around any other point $t = t_0$, as

$$x(t) = \tilde{x}_0 + \tilde{x}_1(t - t_0) + \frac{\tilde{x}_2}{2!}(t - t_0)^2 + \frac{\tilde{x}_3}{3!}(t - t_0)^3 + \cdots \equiv \sum_{n=0}^{\infty} \frac{\tilde{x}_n}{n!}(t - t_0)^n, \tag{1.1.5}$$

with

$$\tilde{x}_n = \left.\frac{d^n x}{dt^n}\right|_{t=t_0} . \tag{1.1.6}$$

For the time being we shall not deal with *non-analytic* functions where such an expansion is not possible.

1.1.2 Small displacements $\Longrightarrow$ Linear response

Surprisingly, we can usually get quite far by approximating a Taylor expansion with its first terms. Of course, the trick is to do this for the right quantity in the appropriate limit, as we shall see shortly. An important example of this in mechanics is the famous *Hooke's law* which states that the force F is proportional (and opposed) to the displacement. Imagine pushing a spring, or any other elastic body, by a displacement x away from its equilibrium state. The spring responds by exerting a force $F(x)$, which can be represented by the Taylor series

$$F(x) = f_0 + f_1 x + \frac{f_2}{2!}x^2 + \cdots \approx -Kx, \tag{1.1.7}$$

where we have used common sense to decide that the first coefficient f_0 is zero (at equilibrium), and the second is negative ($K = -f_1 > 0$). Naturally higher order terms are present in any material, and will invalidate the Hookian approximation when the displacement exceeds say $x^* \approx f_1/f_2$. As long as the contribution from higher order terms in the series is small, which will always be the case for small enough displacements, we can use this approximation. A *linear response* to perturbations is quite frequently used in physics as it is amenable to analytic computations that usually provide much insight. It is, however, important to be aware of the limits to validity of linearized models; the non-linear regime is harder to handle, and could lead to very different behaviors (e.g when a spring breaks).

In mechanical systems, we can relate the force to the derivative of another function, the potential energy $V(x)$ by

$$F(x) = -\frac{dV(x)}{dx}. \tag{1.1.8}$$

Note that if we construct a Taylor series for the potential energy corresponding to displacements *around an equilibrium* point, we get

$$V(x) = V_0 + \frac{K}{2}x^2 + \text{higher order terms}\,. \tag{1.1.9}$$

Quite generally, expansions around an equilibrium position, corresponding to zero force, start with a quadratic term. Ignoring higher order terms leads to a quadratic or *harmonic* potential. The ideal Hookian spring thus has a linear force law, and a harmonic potential energy.

1.1.3 Instantaneous reaction $\Longrightarrow$ Simple differential equations

We noted that displacing a particle from its equilibrium position leads to a response, quantified by the force $F(x)$. The force in turn causes variations of the coordinate in time as $x(t)$, captured by the Taylor series introduced at the beginning. The leading terms in this series are the velocity $x_1 = v = dx/dt \equiv \dot{x}$ and the acceleration $x_2 = a = d^2x/dt^2 \equiv \ddot{x}$. Assuming that the evolution of the coordinate x at each time is constrained by a relation between these time derivatives and the instananeous force, naturally leads to *differential equations*. While the laws of physics encode these variations through Newton's equation of motion, it is instructive to follow an agnostic perspective.

The simplest assumption we can make is that the velocity of the particle at each time is proportional to the force at that time, i.e. $F(\text{at time } t) = F(x \text{ at time } t) = F(x(t))$, leading to the ordinary differential equation (ODE)

$$\dot{x} = \mu F(x). \tag{1.1.10}$$

This is actually a very good description for a particle moving in a viscous fluid like oil, and μ in the above equation is known as the *mobility*. Let us further assume that at $t = 0$, the particle is displaced by a small amount to x_0. To find its position $x(t)$ as a function of time, we need to solve the *first order differential equation*[1]

$$\dot{x} \approx -\mu K x \equiv -\gamma x. \tag{1.1.11}$$

The force has been linearized, and we have introduced the parameter γ with dimensions of inverse time.

1.1.4 The exponential solution

To find the particle position $x(t)$ we now need to solve the *linear differential equation* $\dot{x} = -\gamma x$. One way of doing this is to develop a Taylor series for $x(t)$ around $t = 0$, whose coefficients are derivatives evaluated at $t = 0$. The differential equation allows us to calculate these derivatives easily, giving in particular

$$\frac{dx}{dt}(t = 0) = -\gamma x(0) = -\gamma x_0. \tag{1.1.12}$$

Higher derivatives can be successively related to lower derivatives by taking derivatives of the differential equation, as

$$\frac{d^2x}{dt^2}(0) = -\gamma \frac{dx}{dt}(0) = +\gamma^2 x_0, \tag{1.1.13}$$

and the general term is

$$\frac{d^n x}{dt^n}(0) = -\gamma \frac{d^{n-1}x}{dt^{n-1}}(0) = (-\gamma)^n x_0. \tag{1.1.14}$$

[1]The order of a differential equation is defined by the highest derivative term appearing in the equation, e.g. if $d^n x/dt^n$ is the highest derivative, the ODE is of n^{th} order.

The solution can thus be obtained from the series

$$x(t) = x_0 - \gamma x_0 t + \frac{\gamma^2}{2!} x_0 t^2 + \cdots = x_0 \sum_{n=0}^{\infty} \frac{(-\gamma t)^n}{n!} = x_0 e^{-\gamma t}. \tag{1.1.15}$$

For the final step, we have employed the series expansion for the exponential function.

• *The exponential function has the nice property of keeping its form under differentiation. It thus appears quite generally as a solution to all linear differential equations.*

1.1.5 Time reversal symmetry

The above solution describes a displacement that decays to zero over a characteristic time $\tau = 1/\gamma$. While this may be an acceptable solution to a creature living in viscous oil, a being in free space notices that things typically do not come to rest, but will instead oscillate around their equilibrium position for long times. We then make the crucial observation that an oscillation looks the same going forward or backward in time. If the laws of nature have such *time reversal symmetry* then we should use an equation of motion that respects this, and does not change under $t \to -t$. Since the velocity changes sign under this transformation, the earlier proposed Eq. (**??**) does not describe such a situation. Insisting on time reversal symmetry as a property of nature then leads to Newton's law of motion in which the acceleration, $a = d^2x/dt^2 \equiv \ddot{x}$ (which is invariant under $t \to -t$) is proportional to force, i.e.

$$m\ddot{x} = F(x)\,, \tag{1.1.16}$$

where m is the mass. In the linear regime $F(x) \approx -Kx$, with K as the Hookian coefficient, we thus arrive to the *second order differential equation*

$$\ddot{x} = -\omega_0^2 x\,, \tag{1.1.17}$$

where $\omega_0 = \sqrt{K/m}$ has dimensions of inverse time. This equation describes *Simple Harmonic Oscillations* (SHOs) as shown next.

We can obtain the solution to this equation by the same series method as before. However, while even derivatives at $t = 0$ can be obtained from the initial displacement x_0, the odd derivatives are related to the initial velocity v_0. The complete solution is thus

$$\begin{aligned} x(t) &= x_0 + v_0 t - \frac{\omega_0^2}{2!} x_0 t^2 - \frac{\omega_0^2}{3!} v_0 t^3 + \cdots \\ &= x_0 \left[1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} + \cdots \right] + \frac{v_0}{\omega_0} \left[(\omega_0 t) - \frac{(\omega_0 t)^3}{3!} + \cdots \right] \\ &= x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)\,. \end{aligned} \tag{1.1.18}$$

For the final step we have assumed familiarity with the series expansions for sine and cosine functions. The solution in this case is a periodic function, i.e. it represents oscillations around the origin. The function repeats with a *period* of

$$T = \frac{2\pi}{\omega_0} = \nu^{-1}, \tag{1.1.19}$$

where the *frequency* ν has units of Hertz (inverse second). The parameter ω_0 is the *angular frequency* and has units of radians per second.

• *Note that a solution to the second order differential equation depends on two initial conditions, x_0 and v_0. From the series approach you can see that for n^{th} order differential equations, one needs to specify n initial conditions.*

⋆ *Pendulum:* A mass m at the end of a string of length ℓ is disturbed from equilibrium. Let us denote θ the angle from the vertical. The tangential acceleration is $\ell\ddot{\theta}$, while the tangential component of the restoring force due to gravity is $mg\sin\theta$. From Newton's law we have

$$m\ell\ddot{\theta} = -mg\sin\theta \approx -mg\theta, \tag{1.1.20}$$

where the last step comes from linearizing the force. This can be cast in the standard form $\ddot{\theta} = -\omega_0^2\theta$ for SHO, with angular frequency $\omega_0 = \sqrt{g/\ell}$. Note that the angular frequency is independent of mass (or shape) of the object. This is a physical principle that could not be guessed from mathematics alone, and thus points to existence of yet another symmetry of nature.

(3) A more general equation describing the motion of particle in a fluid includes both inertial and friction terms, taking the form

$$F(x) = m\ddot{x} + \frac{1}{\mu}\dot{x} + \cdots . \tag{1.1.21}$$

While the Newtonian term, $m\ddot{x}$ is time reversal symmetric, inclusion of friction via $\dot{x}/\mu$ removes this symmetry. A linear response around equilibrium then leads to damped oscillations that will be quantified later. Note, however, that in the spirit of a series expansion we can in principle add to the right hand side higher order derivatives, such as d^3x/dt^3, as well as nonlinear terms such as $\ddot{x}\dot{x}^2$. Indeed the former appears as a form of quantum friction in vacuum, while near relativistic speeds allow for the latter; the latter being negligible at ordinary settings dealing with velocities much smaller than the speed of light. The lesson is that even commonly occurring equations of motion may be regarded as expressions of low order terms in a series expansion, and thus as mathematical constructs that transcend particular physics applications.

1.1.6 The Energy Function

For future reference we make the following observations:

(1) Any first order differential equation of the form $\dot{x} = \mu F(x)$ can be cast as

$$\dot{x} = -\mu\frac{dV(x)}{dx}, \quad \text{with} \quad V(x) = -\int^x dx' F(x'). \tag{1.1.22}$$

As its argument changes with time, so does the potential $V(x(t))$, and its variations are obtained using the *chain rule* of differentiation as

$$\frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt} = -\mu\left(\frac{dV}{dx}\right)^2 \leq 0. \tag{1.1.23}$$

For $\mu > 0$, the value of potential can only decrease with time, and Eq. (**??**) describes *gradient descent* in the potential $V(x)$. The coordinate x proceeds towards a stationary (equilibrium) state corresponding to closest minimum of the potential $V(x)$. (The stationary point at a local maximum of $V(x)$ is referred to as an unstable equilibrium point.)

(2) For any second order differential equation of the form $m\ddot{x} = F(x) = -dV(x)/dx$, we can define a first integral by multiplying both sides of the equation with $\dot{x}$, and rearranging as

$$0 = m\dot{x}\ddot{x} + \dot{x}\frac{dV(x)}{dx} = \frac{d}{dt}\left[m\frac{\dot{x}^2}{2} + V(x)\right] \equiv \frac{dE}{dt}\,. \tag{1.1.24}$$

This immediately implies that the quantity

$$E(t) = m\frac{\dot{x}^2}{2} + V(x) = E_0, \tag{1.1.25}$$

is a constant of motion that does not change over time. In the context of a particle, E corresponds to the sum of a kinetic energy $m\dot{x}^2/2$, and a potential energy $V(x)$. For small distortions around an equilibrium position ($x = \dot{x} = 0$), the energy can then be expanded as

$$E(t) = \frac{m}{2}\dot{x}^2 + \frac{K}{2}x^2 + \text{higher order terms} = E_0\,. \tag{1.1.26}$$

Conservation of energy then leads to

$$0 = \frac{dE}{dt} \approx M\dot{x}\ddot{x} + Kx\dot{x} = \dot{x}\left(M\ddot{x} + Kx\right). \tag{1.1.27}$$

Setting the term in the brackets to zero reproduces the equation of motion, in this case again describing SHOs with $\omega_0 = \sqrt{K/M}$.

Recap

We encountered the following *linear* differential equations:

- *(i)* First order (simple damping):

$$\dot{x} = -\gamma x, \qquad \Rightarrow \qquad x(t) = x_0 e^{-\gamma t}. \tag{1.1.28}$$

- *(ii)* Second order (SHO):

$$\ddot{x} = -\omega_0^2 x, \qquad \Rightarrow \qquad x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\,. \tag{1.1.29}$$

Note that the above second equation has two classes of solutions: (a) $x_0\cos(\omega_0 t)$, corresponding to an initial displacement x_0 with zero velocity at $t = 0$; and (b) $v_0\sin(\omega_0 t)/\omega_0$, which describes starting the particle from the origin, with velocity v_0 at $t = 0$. The general solution is simply the sum of the two cases, i.e. if we want to find the displacement of a particle launched at x_0 with velocity v_0, we simply need to add the solutions in (a), and (b). This is a simple example of the very important *superposition principle* which is an important property of linear systems that will be discussed later.