3.2 Solving PDEs

3.2.1 Separable solutions

There are several methods for obtaining solutions to partial differential equations. One way, which is physically close to searching for normal modes, is to look for *separable solutions*, in which u is a product of two functions that depend on x and t separately, i.e.

$$u(x,t) = X(x)T(t)\,. \tag{3.2.1}$$

Substituting this form in Eq. (3.1.14), $\eta\dot{u} + \rho\ddot{u} = -Ju + u''$, and dividing by $u = XT$, leads to

$$\frac{\rho\ddot{T} + \eta\dot{T} + JT}{T} = K\frac{X''}{X}\,. \tag{3.2.2}$$

The left hand side is now only a function of time t, while the right hand side depends only on x. The only way that a function of x can always be equal to a function of t is if both sides are constants, e.g. equal to λ. This requirement reduces the problem to two ordinary differential equations

$$\rho\ddot{T} + \eta\dot{T} + JT = \lambda T\,, \tag{3.2.3}$$

and

$$KX'' = \lambda X\,. \tag{3.2.4}$$

We could have placed the term proportional to J with either equation; the current choice makes analogy to normal modes of a chain more transparent. Anticipating such normal modes, we look for solutions of the form

$$X(x) \propto \sin(kx + \theta)\,, \tag{3.2.5}$$

which require $\lambda = -Kk^2$. The *wave-number* k determines the frequency of repetitions of the displacement along x, which recur every *wave-length*

$$\lambda = \frac{2\pi}{k}\,. \tag{3.2.6}$$

The ODE for $T(t)$ resembles that of a damped harmonic oscillator, suggesting trial solutions in the form of a complex exponential

$$T(t) \propto e^{i\omega t}\,. \tag{3.2.7}$$

Substituting this into Eq. (3.2.3) leads to the condition

$$-\rho\omega^2 + i\eta\omega + J = \lambda = -Kk^2\,. \tag{3.2.8}$$

Thus the assumed separable form is a valid solution as long as the complex frequency ω and the wave-vector k are related as in Eq. (3.2.8). The resulting function, $\omega(k)$, is an important intrinsic property of the linear PDE, and is referred to as the *dispersion relation.* The separable solution can be interpreted as representing a normal mode of the system with wavenumber k, whose dynamics are governed by a complex frequency $\omega(k)$.

- For the *diffusion equation*, with $\rho = J = 0$ and setting $K/\eta = D$ in Eq. (3.2.8), the dispersion relation is complex with $\omega(k) = iDk^2$.
- For the *wave equation*, with $\eta = J = 0$ and setting $K/\rho = v^2$ in Eq. (3.2.8), the dispersion relation is linear with $\omega(k) = vk$.