### 3.2.2 Quantized modes

In fact, to determine the exact normal modes of a system, we have to also incorporate its boundary conditions. This leads to restrictions on the allowed values of $k$, as in the following examples for $x$ in the interval from 0 to $L$.

- **Closed on both ends:** In this case, Eq. (3.2.5) must be constrained such that

$$u(x=0,t)=u(x=L,t)=0, \quad \Rightarrow \quad \sin(\theta)=\sin(kL+\theta)=0\,. \tag{3.2.9}$$

  The first condition is satisfied by $\theta=0$, while the second condition requires

$$\sin(kL)=0, \quad \Rightarrow \quad kL=n\pi\,. \tag{3.2.10}$$

  We thus obtain a discrete set of normal modes $X(x)\propto\sin(k_n x)$, with

$$k_n=\frac{n\pi}{L}, \qquad \text{for} \quad n=1,2,3,\cdots\,. \tag{3.2.11}$$

  Note that the wave-length of the $n$th mode is $\lambda_n=2L/n$.

- **Open at both ends:** The boundary conditions are

$$u'(x=0,t)=u'(x=L,t)=0, \quad \Rightarrow \quad \cos(\theta)=\cos(kL+\theta)=0\,. \tag{3.2.12}$$

  The first condition implies, $\theta=\pi/2$, i.e. normal mode solutions of the form $X(x)\propto\cos(k_n x)$. The second boundary condition again restricts $k_n$ to integer multiples of $\pi$, i.e.

$$k_n=\frac{n\pi}{L}, \qquad \text{for} \quad n=0,1,2,3,\cdots\,. \tag{3.2.13}$$

  However, in this case $n=0$ is an acceptable solution. It describes a normal mode in which the whole system is translated as a single body.

- **Periodic system:** The requirement of

$$u(x=0,t)=u(x=L,t)=0, \quad \Rightarrow \quad \sin(\theta)=\sin(kL+\theta)\,, \tag{3.2.14}$$

  is satisfied by any $k_n$ which is a multiple of $2\pi$, i.e. for

$$k_n=\frac{2\pi n}{L}, \qquad \text{for} \quad n=0,1,2,3,\cdots\,, \tag{3.2.15}$$

  and corresponding wave-lengths of $\lambda_n=L/n$. Note that in this case each normal mode is two fold *degenerate*, i.e. $X(x)\propto\sin(k_n x)$ and $X(x)\propto\cos(k_n x)$ have exactly the same frequency.

- **Open at one end, and closed at the other:** The closed end boundary condition $u(x=0,t)=0$ can be satisfied by choosing $X(x)=A\sin(kx)$. The boundary condition at the open end yields

$$u'(x=L)=0, \quad \Rightarrow \quad \cos(kL)=0\,, \tag{3.2.16}$$

  whose solutions are

$$k_n=\left(n+\frac{1}{2}\right)\pi, \qquad \text{for} \quad n=0,1,2,\cdots\,. \tag{3.2.17}$$