3.2.3 Superposition of normal modes

The normal modes provide specific, and particularly simple solutions to the PDE. Since we are dealing with a linear PDE, other solutions can be obtained by superposing normal modes. In fact, the full set of normal modes provides a *complete basis* for the space of functions on the given domain, such that *any solution* to the linear PDE, *with appropriate boundary conditions*, can be decomposed into a superposition of normal modes.

To illustrate this, let us consider the wave equation, with dispersion relation $\omega = vk$, in case of a string that is fixed at both ends. Any solution of the wave-equation must thus satisfy $u(0,t) = u(L,t) = 0$, and can be expressed by a sum of normal modes as

$$u(x,t) = \sum_{n=1}^{\infty} \mathcal{A}_n \sin(k_n x) \sin(\omega_n t + \theta_n) \equiv \sum_{n=1}^{\infty} \sin(k_n x) \left[A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right] , \tag{3.2.18}$$

where $k_n = n\pi/L$ for integer n, and $\omega_n = vk_n$. We have also indicated two equivalent forms for the time variations of the normal modes that are sometimes useful.

The coefficients $\{\mathcal{A}_n, \theta_n\}$ or $\{A_n, B_n\}$ can for example be fixed by giving the initial conditions $u(x, t = 0)$ and $\dot{u}(x, t = 0)$. The velocity of the string is obtained as

$$\begin{aligned} \dot{u}(x,t) &= \sum_{n=1}^{\infty} \mathcal{A}_n \omega_n \sin(k_n x) \cos(\omega_n t + \theta_n) \\ &\equiv \sum_{n=1}^{\infty} \sin(k_n x) \left[\omega_n A_n \cos(\omega_n t) - \omega_n B_n \sin(\omega_n t)\right] , \end{aligned} \tag{3.2.19}$$

and hence the initial conditions can be written as

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) , \quad \text{and} \quad \dot{u}(x,0) = \sum_{n=1}^{\infty} \omega_n A_n \sin\left(\frac{n\pi x}{L}\right) . \tag{3.2.20}$$

The coefficients $\{A_n\}$ and $\{B_n\}$ can now be obtained from $\dot{u}(x,0)$ and $u(x,0)$ respectively, using the following result[2]

$$\int_0^L dx \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \left\{\begin{array}{cc} 0 & \text{for } n \neq m \\ L/2 & \text{for } n = m \end{array}\right\} \equiv \frac{L}{2}\delta_{nm} . \tag{3.2.21}$$

Multiplying each side of the series for initial conditions in Eq. (3.2.20) by $\sin\left(\frac{m\pi x}{L}\right)$, and integrating over x, we obtain

$$\left\{\begin{array}{l} \displaystyle\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) u(x,0) = \frac{L}{2} B_m \\ \displaystyle\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \dot{u}(x,0) = \frac{L}{2} \omega_m A_m \end{array}\right. , \Rightarrow \quad \left\{\begin{array}{l} \displaystyle B_m = \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) u(x,0) \\ \displaystyle A_m = \frac{2}{\omega_m L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \dot{u}(x,0) \end{array}\right. . \tag{3.2.22}$$

[2]This is usually referred to as an *orthogonality condition* for the normal mode basis.