3.2.4 Plucked String

As a specific example, consider a string that is plucked at its mid-point, and then released, so that the initial conditions are given by

$$u(x,t=0) = \begin{cases} \dfrac{2wx}{L} & \text{for} \quad 0 \le x \le L/2 \\ \dfrac{2w(L-x)}{L} & \text{for} \quad L/2 \le x \le L \end{cases}, \qquad \text{and} \qquad \dot{u}(x,t=0) = 0\,. \tag{3.2.23}$$

According to the above general result, $A_n = 0$ immediately follows from $\dot{u}(x,0) = 0$, while

$$B_n = \frac{4w}{L^2}\left\{\int_0^{L/2} dx x \sin\left(\frac{n\pi x}{L}\right) + \int_{L/2}^{L} dx (L-x)\sin\left(\frac{n\pi x}{L}\right)\right\} \tag{3.2.24}$$

$$= \frac{4w}{L^2}\left\{ -\frac{L}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/2}\right. \tag{3.2.25}$$

$$\left. -\frac{L}{n\pi}(L-x)\cos\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L}\right\} \tag{3.2.26}$$

$$= \frac{4w}{L^2}\left\{ -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right\} \tag{3.2.27}$$

$$= \frac{8w}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right). \tag{3.2.28}$$

We see that all the even terms in the series (also known as even harmonics) are absent. The odd terms alternate in sign, and diminish in magnitude as $1/n^2$. Using the above result we can reconstruct the full time dependence of the shape of the string as

$$u(x,t) = \frac{8w}{\pi^2}\sum_{\text{odd } n}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) \tag{3.2.29}$$

$$= \frac{8w}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi vt}{L}\right) + \frac{1}{16}\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4\pi vt}{L}\right) + \cdots\right].$$