3.2 Solving PDEs

3.2.1 Separable solutions

There are several methods for obtaining solutions to partial differential equations. One way, which is physically close to searching for normal modes, is to look for *separable solutions*, in which u is a product of two functions that depend on x and t separately, i.e.

$$u(x,t) = X(x)T(t)\,. \tag{3.2.1}$$

Substituting this form in Eq. (3.1.14), $\eta\dot{u} + \rho\ddot{u} = -Ju + u''$, and dividing by $u = XT$, leads to

$$\frac{\rho\ddot{T} + \eta\dot{T} + JT}{T} = K\frac{X''}{X}\,. \tag{3.2.2}$$

The left hand side is now only a function of time t, while the right hand side depends only on x. The only way that a function of x can always be equal to a function of t is if both sides are constants, e.g. equal to λ. This requirement reduces the problem to two ordinary differential equations

$$\rho\ddot{T} + \eta\dot{T} + JT = \lambda T\,, \tag{3.2.3}$$

and

$$KX'' = \lambda X\,. \tag{3.2.4}$$

We could have placed the term proportional to J with either equation; the current choice makes analogy to normal modes of a chain more transparent. Anticipating such normal modes, we look for solutions of the form

$$X(x) \propto \sin(kx + \theta)\,, \tag{3.2.5}$$

which require $\lambda = -Kk^2$. The *wave-number* k determines the frequency of repetitions of the displacement along x, which recur every *wave-length*

$$\lambda = \frac{2\pi}{k}\,. \tag{3.2.6}$$

The ODE for $T(t)$ resembles that of a damped harmonic oscillator, suggesting trial solutions in the form of a complex exponential

$$T(t) \propto e^{i\omega t}\,. \tag{3.2.7}$$

Substituting this into Eq. (3.2.3) leads to the condition

$$-\rho\omega^2 + i\eta\omega + J = \lambda = -Kk^2\,. \tag{3.2.8}$$

Thus the assumed separable form is a valid solution as long as the complex frequency ω and the wave-vector k are related as in Eq. (3.2.8). The resulting function, $\omega(k)$, is an important intrinsic property of the linear PDE, and is referred to as the *dispersion relation*. The separable solution can be interpreted as representing a normal mode of the system with wavenumber k, whose dynamics are governed by a complex frequency $\omega(k)$.

- For the *diffusion equation*, with $\rho = J = 0$ and setting $K/\eta = D$ in Eq. (3.2.8), the dispersion relation is complex with $\omega(k) = iDk^2$.
- For the *wave equation*, with $\eta = J = 0$ and setting $K/\rho = v^2$ in Eq. (3.2.8), the dispersion relation is linear with $\omega(k) = vk$.

3.2.2 Quantized modes

In fact, to determine the exact normal modes of a system, we have to also incorporate its boundary conditions. This leads to restrictions on the allowed values of k, as in the following examples for x in the interval from 0 to L.

- **Closed on both ends:** In this case, Eq. (3.2.5) must be constrained such that

$$u(x=0,t) = u(x=L,t) = 0, \quad \Rightarrow \quad \sin(\theta) = \sin(kL+\theta) = 0\,. \tag{3.2.9}$$

  The first condition is satisfied by $\theta = 0$, while the second condition requires

$$\sin(kL) = 0, \quad \Rightarrow \quad kL = n\pi\,. \tag{3.2.10}$$

  We thus obtain a discrete set of normal modes $X(x) \propto \sin(k_n x)$, with

$$k_n = \frac{n\pi}{L}, \qquad \text{for} \quad n = 1, 2, 3, \cdots\,. \tag{3.2.11}$$

  Note that the wave-length of the nth mode is $\lambda_n = 2L/n$.

- **Open at both ends:** The boundary conditions are

$$u'(x=0,t) = u'(x=L,t) = 0, \quad \Rightarrow \quad \cos(\theta) = \cos(kL+\theta) = 0\,. \tag{3.2.12}$$

  The first condition implies, $\theta = \pi/2$, i.e. normal mode solutions of the form $X(x) \propto \cos(k_n x)$. The second boundary condition again restricts k_n to integer multiples of π, i.e.

$$k_n = \frac{n\pi}{L}, \qquad \text{for} \quad n = 0, 1, 2, 3, \cdots\,. \tag{3.2.13}$$

  However, in this case $n = 0$ is an acceptable solution. It describes a normal mode in which the whole system is translated as a single body.

- **Periodic system:** The requirement of

$$u(x=0,t) = u(x=L,t) = 0, \quad \Rightarrow \quad \sin(\theta) = \sin(kL+\theta)\,, \tag{3.2.14}$$

  is satisfied by any k_n which is a multiple of 2π, i.e. for

$$k_n = \frac{2\pi n}{L}, \qquad \text{for} \quad n = 0, 1, 2, 3, \cdots\,, \tag{3.2.15}$$

  and corresponding wave-lengths of $\lambda_n = L/n$. Note that in this case each normal mode is two fold *degenerate*, i.e. $X(x) \propto \sin(k_n x)$ and $X(x) \propto \cos(k_n x)$ have exactly the same frequency.

- **Open at one end, and closed at the other:** The closed end boundary condition $u(x = 0, t) = 0$ can be satisfied by choosing $X(x) = A\sin(kx)$. The boundary condition at the open end yields

$$u'(x = L) = 0, \quad \Rightarrow \quad \cos(kL) = 0\,, \tag{3.2.16}$$

whose solutions are

$$k_n = \left(n + \frac{1}{2}\right)\pi, \qquad \text{for} \quad n = 0, 1, 2, \cdots\,. \tag{3.2.17}$$

3.2.3 Superposition of normal modes

The normal modes provide specific, and particularly simple solutions to the PDE. Since we are dealing with a linear PDE, other solutions can be obtained by superposing normal modes. In fact, the full set of normal modes provides a *complete basis* for the space of functions on the given domain, such that *any solution* to the linear PDE, *with appropriate boundary conditions*, can be decomposed into a superposition of normal modes.

To illustrate this, let us consider the wave equation, with dispersion relation $\omega = vk$, in case of a string that is fixed at both ends. Any solution of the wave-equation must thus satisfy $u(0, t) = u(L, t) = 0$, and can be expressed by a sum of normal modes as

$$u(x,t) = \sum_{n=1}^{\infty} \mathcal{A}_n \sin\left(k_n x\right)\sin\left(\omega_n t + \theta_n\right) \equiv \sum_{n=1}^{\infty} \sin\left(k_n x\right)\left[A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right]\,, \tag{3.2.18}$$

where $k_n = n\pi/L$ for integer n, and $\omega_n = vk_n$. We have also indicated two equivalent forms for the time variations of the normal modes that are sometimes useful.

The coefficients $\{\mathcal{A}_n, \theta_n\}$ or $\{A_n, B_n\}$ can for example be fixed by giving the initial conditions $u(x, t = 0)$ and $\dot{u}(x, t = 0)$. The velocity of the string is obtained as

$$\begin{aligned} \dot{u}(x,t) &= \sum_{n=1}^{\infty} \mathcal{A}_n \omega_n \sin\left(k_n x\right)\cos\left(\omega_n t + \theta_n\right) \\ &\equiv \sum_{n=1}^{\infty} \sin\left(k_n x\right)\left[\omega_n A_n \cos(\omega_n t) - \omega_n B_n \sin(\omega_n t)\right]\,, \end{aligned} \tag{3.2.19}$$

and hence the initial conditions can be written as

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\,, \quad \text{and} \quad \dot{u}(x,0) = \sum_{n=1}^{\infty} \omega_n A_n \sin\left(\frac{n\pi x}{L}\right)\,. \tag{3.2.20}$$

The coefficients $\{A_n\}$ and $\{B_n\}$ can now be obtained from $\dot{u}(x, 0)$ and $u(x, 0)$ respectively, using the following result[2]

$$\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \left\{\begin{array}{cc} 0 & \text{for } n \neq m \\ L/2 & \text{for } n = m \end{array}\right\} \equiv \frac{L}{2}\delta_{nm}\,. \tag{3.2.21}$$

[2]This is usually referred to as an *orthogonality condition* for the normal mode basis.

Multiplying each side of the series for initial conditions in Eq. (3.2.20) by $\sin\left(\frac{m\pi x}{L}\right)$, and integrating over x, we obtain

$$\begin{cases} \displaystyle\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) u(x,0) = \frac{L}{2} B_m \\ \displaystyle\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \dot{u}(x,0) = \frac{L}{2}\omega_m A_m \end{cases}, \Rightarrow \quad \begin{cases} \displaystyle B_m = \frac{2}{L}\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) u(x,0) \\ \displaystyle A_m = \frac{2}{\omega_m L}\int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \dot{u}(x,0) \end{cases}. \tag{3.2.22}$$

3.2.4 Plucked String

As a specific example, consider a string that is plucked at its mid-point, and then released, so that the initial conditions are given by

$$u(x, t=0) = \begin{cases} \dfrac{2wx}{L} & \text{for} \quad 0 \le x \le L/2 \\ \dfrac{2w(L-x)}{L} & \text{for} \quad L/2 \le x \le L \end{cases}, \qquad \text{and} \qquad \dot{u}(x, t=0) = 0\,. \tag{3.2.23}$$

According to the above general result, $A_n = 0$ immediately follows from $\dot{u}(x,0) = 0$, while

$$B_n = \frac{4w}{L^2}\left\{\int_0^{L/2} dx x \sin\left(\frac{n\pi x}{L}\right) + \int_{L/2}^L dx (L-x) \sin\left(\frac{n\pi x}{L}\right)\right\} \tag{3.2.24}$$

$$= \frac{4w}{L^2}\left\{ -\frac{L}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/2} \right. \tag{3.2.25}$$

$$\left. -\frac{L}{n\pi}(L-x)\cos\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L} \right\} \tag{3.2.26}$$

$$= \frac{4w}{L^2}\left\{ -\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right\} \tag{3.2.27}$$

$$= \frac{8w}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right). \tag{3.2.28}$$

We see that all the even terms in the series (also known as even harmonics) are absent. The odd terms alternate in sign, and diminish in magnitude as $1/n^2$. Using the above result we can reconstruct the full time dependence of the shape of the string as

$$u(x,t) = \frac{8w}{\pi^2}\sum_{\text{odd } n}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right) \tag{3.2.29}$$

$$= \frac{8w}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi v t}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi v t}{L}\right) + \frac{1}{16}\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4\pi v t}{L}\right) + \cdots\right].$$

Recap

- Separable solutions to linear PDEs of the form

$$\eta\dot{u} + \rho\ddot{u} = -Ju + Ku'' \,, \tag{3.2.30}$$

  are closely related to normal modes of the Laplacian operator, and can take the form

$$u(x,t) \propto e^{i\omega(k)t} \sin(kx+\theta) \,, \tag{3.2.31}$$

  with $\omega(k)$ given by the dispersion relation, as solution to

$$i\omega\eta - \omega^2\rho + Ju = -Kk^2 \,. \tag{3.2.32}$$

- In a finite system, the allowed values of k are determined by boundary conditions. For example, with closed boundary conditions $u(0,t) = u(L,t) = 0$, $k_n = n\pi/L$ for $n = 1, 2, 3, \cdots$.