3.3 Fourier analysis

3.3.1 Fourier Series

The procedure for decomposing the initial condition as a sum of terms proportional to $\sin(n\pi x/L)$ is an example of *Fourier transformation.* In fact, one can similarly obtain Fourier series for any function defined on any interval. The choice of boundary conditions is part of indicating the interval under consideration. Let us consider *periodic boundary conditions*,also describing functions $f(x) = f(x+L)$ that repeat cyclically upon translation by L. Such functions can be constructed by superposition of sine and cosine functions of the same period, as

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right)\right] . \tag{3.3.1}$$

The sine and cosine functions are orthogonal in the interval, in the sense that they obey the relations

$$\begin{cases} \int_0^L dx \cos\left(\frac{2\pi m x}{L}\right) \cos\left(\frac{2\pi n x}{L}\right) = \frac{L}{2}\delta_{mn} \\ \int_0^L dx \sin\left(\frac{2\pi m x}{L}\right) \sin\left(\frac{2\pi n x}{L}\right) = \frac{L}{2}\delta_{mn} \\ \int_0^L dx \sin\left(\frac{2\pi m x}{L}\right) \cos\left(\frac{2\pi n x}{L}\right) = 0 \end{cases} , \tag{3.3.2}$$

as

$$\begin{cases} A_0 = \frac{1}{L}\int_0^L dx f(x) \\ A_n = \frac{2}{L}\int_0^L dx f(x) \cos\left(\frac{2\pi n x}{L}\right) \\ B_n = \frac{2}{L}\int_0^L dx f(x) \sin\left(\frac{2\pi n x}{L}\right) \end{cases} . \tag{3.3.3}$$

Another perspective is to scale the interval from $x \in [0, 2\pi]$ to an angle $\theta = 2\pi x/L \in [0, 2\pi]$.The appropriate normal modes (basis functions) for such periodicity are $\cos(n\theta)$ for $n = 0, 1, 2, \cdots$, and $\sin(n\theta)$ for $n = 1, 2, \cdots$. We can then write for any periodic function

$$f(\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] . \tag{3.3.4}$$

The coefficients in the expansion can be obtained using the following orthogonality relations

$$\begin{cases} \displaystyle\int_0^{2\pi} d\theta \cos(m\theta)\cos(n\theta) = \pi\delta_{mn} \\ \displaystyle\int_0^{2\pi} d\theta \sin(m\theta)\sin(n\theta) = \pi\delta_{mn} \\ \displaystyle\int_0^{2\pi} d\theta \sin(m\theta)\cos(n\theta) = 0 \end{cases}, \tag{3.3.5}$$

as

$$\begin{cases} A_0 = \displaystyle\frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta) \\ A_n = \displaystyle\frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta)\cos(n\theta) \\ B_n = \displaystyle\frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta)\sin(n\theta) \end{cases}. \tag{3.3.6}$$

As a first example, consider a "square wave," i.e. a function

$$f_1(\theta) = \begin{cases} 1 & \text{for } 0 < \theta < \pi \\ -1 & \text{for } \pi < \theta < 2\pi \end{cases}. \tag{3.3.7}$$

The coefficients of the Fourier series are

$$A_n = \frac{1}{\pi}\left[\int_0^{\pi} d\theta \cos(n\theta) - \int_{\pi}^{2\pi} d\theta \cos(n\theta)\right] = 0\,, \tag{3.3.8}$$

and

$$B_n = \frac{1}{\pi}\left[\int_0^{\pi} d\theta \sin(n\theta) - \int_{\pi}^{2\pi} d\theta \sin(n\theta)\right] = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]\,. \tag{3.3.9}$$

Hence we can construct a square wave from

$$f_1(\theta) = \frac{4}{\pi}\left[\sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \cdots\right]. \tag{3.3.10}$$

The square wave is certainly not continuous at the boundaries of the interval $[0, 2\pi]$, but the Fourier decomposition still works. Note that using periodicity of 2π, $f_1(-\theta) = -f_1(\theta)$ is an odd function of angle, and hence only the sine terms contribute to the Fourier series.

The procedure can be extended to any interval $x \in [a, b]$. As a second example, consider

$$f_2(x) = x, \qquad \text{for } -L/2 < x < +L/2\,. \tag{3.3.11}$$

The appropriate functions for this interval are $\cos(2n\pi x/L)$ and $\sin(2n\pi x/L)$. Since the function $f_2(x)$ is *odd*, i.e. $f_2(-x) = -f_2(x)$, all the coefficients of the even functions, $\cos(2n\pi x/L)$, are zero. The coefficient of $\sin(2n\pi x/L)$ is given by

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} dx x \sin\left(\frac{2n\pi x}{L}\right) = \frac{1}{n\pi}\left[-x\cos\left(\frac{2n\pi x}{L}\right) + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right|_{-L/2}^{L/2} \quad (3.3.12)$$

$$= -\frac{L}{n\pi}\cos(n\pi)\ . \quad (3.3.13)$$

The Fourier series for this function is thus

$$f_2(x) = \frac{L}{\pi}\left[\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{8\pi x}{L}\right) + \cdots\right], \quad (3.3.14)$$

and differs from that of the square wave only by including the even harmonics with opposite sign!