3.3.2 Complex Exponentials

We can write the Fourier series in more compact form by using the complex exponential representation of sine and cosine, as follows

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left[A_n \frac{e^{2\pi inx/L} + e^{-2\pi inx/La}}{2} + B_n \frac{e^{2\pi inx/L} - e^{-2\pi inx/L}}{2i} \right] \equiv \sum_{-\infty}^{\infty} e^{2\pi inx/L} \tilde{f}_n \,. \tag{3.3.15}$$

Using our previous results for $\{A_n\}$ and $\{B_n\}$, we obtain ($n \geq 0$)

$$\tilde{f}_n = \frac{1}{2}(A_n - iB_n) = \frac{1}{L}\int_0^L dx f(x) \left[\cos\left(\frac{2\pi nx}{L}\right) - i \sin\left(\frac{2\pi nx}{L}\right) \right] = \int_0^L \frac{dx}{L} e^{-2\pi inx/L} f(x) \,. \tag{3.3.16}$$

(It is easy to check that the above form also gives the correct result for $\tilde{f}_0 \equiv A_0$, and for $\tilde{f}_n = A_n + iB_n$.)[3]

If the interval is scaled to map to an angle $\theta = 2\pi x/L$, the above expressions simplify to

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n \frac{e^{in\theta} + e^{-in\theta}}{2} + B_n \frac{e^{in\theta} - e^{-in\theta}}{2i} \right] \equiv \sum_{-\infty}^{\infty} e^{in\theta} \tilde{f}_n \,, \tag{3.3.17}$$

with the inverse relations

$$\tilde{f}_n = \frac{1}{2}(A_n - iB_n) = \frac{1}{2\pi}\int_0^{2\pi} d\theta f(\theta) \left(\cos(n\theta) - i\sin(n\theta)\right) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} f(\theta) \,. \tag{3.3.18}$$

The Fourier components $\tilde{f}_n$ can also be obtained directly from the *orthogonality conditions*

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(m-n)\theta} = \delta_{mn} \,, \tag{3.3.19}$$

where δ_{mn} is the Kronecker delta function introduced earlier.

[3] There is no double counting: in considering sine and cosine modes only positive integers are included, while both positive and negative integers are allowed for $\tilde{f}_n = \tilde{f}^*_{-n}$.