3.3.3 Fourier Integrals

It is even more useful to construct Fourier transformations for functions defined on an entire line $-\infty < x < +\infty$. One way to do so is let the length of the interval L go to infinity. Recall that previously the Fourier series involved sums over exponentials e^{ikx} with allowed values of k discretized (to enforce periodicity) to multiples of $2\pi/L$, as $k_n = 2\pi n/L$. Thus as $L \to \infty$ the distance between discretized points shrinks, resembling the continuous interval $-\infty < k < +\infty$.

The next step is to replace the sum $\sum_n \tilde{f}_n d^{ik_n x}$ in Eq. (3.3.15) with an integral over k. In doing so, note that an integral segment of size dk includes $\frac{dk}{2\pi/L}$ points, and thus

$$f(x) = \sum_n \tilde{f}_n e^{ik_n x} \to \int_{-\infty}^{\infty} \frac{dk}{2\pi/L} \tilde{f}_n e^{ik_n x} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}\,, \tag{3.3.20}$$

where we have defined

$$\tilde{f}(k) = \lim_{L\to\infty} L\tilde{f}_{n=Lk/2\pi}\,. \tag{3.3.21}$$

The inverse relation of Eq. (3.3.16) now takes the form

$$\tilde{f}(k) = L\tilde{f}_n = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}\,. \tag{3.3.22}$$

Thus, any function $f(x)$ can be represented as

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k)\,, \tag{3.3.23}$$

where the *Fourier transform* of the function is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x)\,. \tag{3.3.24}$$

The key to this transformation is the *orthogonality relation*

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')\,, \tag{3.3.25}$$

where $\delta(x-a)$ known as the *Dirac delta-function* is a limiting function highly peaked at $x = a$, and zero every where else, such that

$$\int dx \delta(x-a) g(x) = g(a)\,, \tag{3.3.26}$$

for any function $g(x)$. Comparison of Eqs. (3.3.19) and (3.3.25) indicates the close correspondence between the discrete (Kronecker) and continuous (Dirac) delta-functions. However, whereas δ_{mn} is dimensionless, $\delta(x-a)$ is a density and carries inverse dimensions of x. Thinking of the continuous delta-function as a density helps to understand why its value is infinite when its argument is zero: a unit value has to be obtained by integrating this density over an infinitesimal interval.