3.3.4 Diffusion

Consider density $n(x,t)$ that evolves according to the diffusion equation,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}\,, \tag{3.3.19}$$

starting with an initial condition which is entirely concentrated at the origin $x = 0$, as represented by $n(x,0) = N\delta(x)$. The Dirac delta function is quite frequently used as a *source* in linear PDEs. The evolution of density in time is obtained as follows:

- The initial condition is decomposed in terms of its Fourier components; from Eq. (3.3.16), we find

$$\tilde{n}(k,0) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikx}n(x,0) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikx}N\delta(x) = \frac{N}{2\pi}\,. \tag{3.3.20}$$

- Noting that the general separable solution had time dependence $e^{i\omega t}$, from Eqs. (3.2.7) and (3.2.8), and the dispersion relation $i\omega(k) = -Dk^2$, each Fourier component evolves in time as

$$\tilde{n}(k,t) = e^{i\omega(k)t}\tilde{n}(k,0) = e^{-Dk^2t}\tilde{n}(k,0) = \frac{N}{2\pi}e^{-Dk^2t}\,. \tag{3.3.21}$$

- The solution in x basis at time t is obtained by inverting the Fourier transformation, an in Eq.(3.3.16), resulting in

$$n(x,t) = \int_{-\infty}^{\infty} dx\; e^{ikx}\tilde{n}(k.t) = \frac{N}{2\pi}\int_{-\infty}^{\infty} dx\; e^{ikx-Dk^2t}\,. \tag{3.3.22}$$

- Finally, using the Fourier transform of a *Gaussian* function (see below), we arrive at

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right)\,. \tag{3.3.23}$$

The initial density does spreads over time into a bell-shaped (Gaussian) form centered at the origin, but distributed over a distance that grows as $\sqrt{Dt}$. At the same time, the peak heigh of density decreases as $1/\sqrt{Dt}$, and the integral over the x remains at the initial value of N. It is also interesting to gain another perspective on the Dirac delta-function as the $t \to 0$ limit of a normalized Gaussian of width proportional to t.

At the very beginning of this course, we encountered the exponential function as the solution to the ODE $f'(x) = -\gamma f(x)$. It is interesting to note that all the explicit solutions that we have presented since have involved the exponential in one form or another, e.g. as in sums forming trigonometric functions.The Gaussian in Eq. (3.3.23) is the first truly independent form that we have seen, coming as a solution to the diffusion equation. It is very important to remember the following properties of Gaussian functions:

- A normalized Gaussian function,

$$G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]\,; \tag{3.3.24}$$

  a bell shaped curve peaked at $x = x_0$ that spreads symmetrically around x_0 over a characteristic width of order σ.

- The included prefactor ensures that $\int_{-\infty}^{\infty} dx\, G(x) = 1$, and that it can thus be regarded as a probability density. Using this weight, we find

$$\int_{-\infty}^{\infty} dx\; x\; G(x) = x_0\,, \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx\; (x-x_0)^2\; G(x) = \sigma^2\,. \tag{3.3.25}$$

- The Fourier transform of a Gaussian is a similar (albeit not normalized) Gaussian

$$\int_{-\infty}^{\infty} dx\; e^{ikx}\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] = \exp\left[-ikx_0 - \frac{\sigma^2 k^2}{2}\right]\,. \tag{3.3.26}$$