3.3.5 Wave packet

Let us next use Fourier integrals to obtain the solution to the wave equation

$$\frac{\partial^2 h}{\partial t^2} = v^2 \frac{\partial^2 h}{\partial x^2}\,, \tag{3.3.27}$$

starting from an initial *stationary* Gaussian shape

$$h(x, t=0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \text{with} \qquad \dot{h}(x, t=0) = 0\,. \tag{3.3.28}$$

We can decompose this initial shape into a Fourier integral

$$h(x, 0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{h}(k, 0)\,, \tag{3.3.29}$$

which we the use of Eq. (3.3.26) yields

$$\tilde{h}(k, 0) = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-ikx - \frac{x^2}{2\sigma^2}\right) = \exp\left(-\frac{\sigma^2 k^2}{2}\right)\,. \tag{3.3.30}$$

The dispersion relation for the wave equation $\omega^2 = v^2 k^2$ has two branches with $\omega = \pm vk$. For mode of wavenumber k, time evolution of the form

$$\tilde{h}(k, t) = \tilde{h}(k, 0)\left[f(k)e^{ikvt} + (1 - f(k))e^{-ikvt}\right]\,, \tag{3.3.31}$$

is a valid solution for any $f(k)$. The choice of $f(k)$ is dictated by the initial condition. In this case $\dot{h}(x, t=0) = 0$, requires $\dot{\tilde{h}}(k, t=0) = 0$, i.e.

$$\dot{\tilde{h}}(k, t) = \tilde{h}(k, 0)\left[ikvf(k)e^{ikvt} - ikv(1 - f(k))e^{-ikvt}\right] = 0 \quad \Rightarrow \quad f(k) = \frac{1}{2}\,. \tag{3.3.32}$$

This leads to

$$\begin{aligned} h(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) e^{ikx} \cos(vkt) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) e^{ikx} \frac{e^{ikvt} + e^{-ikvt}}{2} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) \left(e^{ik(x+vt)} + e^{ik(x-vt)}\right) \\ &= \frac{1}{2}\left[h(x + vt, 0) + h(x - vt, 0)\right]\,. \end{aligned} \tag{3.3.33}$$

This solution describes two Gaussian bumps, each half in size, one moving to the left, and the other to the right with velocity v. As long as these bumps are far away from any boundaries they maintain their shapes. Each bump represents a *traveling wave*, a disturbance that propagates without changing its shape.

The above example indicates that the wave equation admits solutions very different from the separable solutions that are also known as *standing waves*. There is in fact another procedure for solving partial differential equations, known as *the method of characteristics*, which naturally yields travelling wave solutions. In this method one searches for *characteristic curves* in the (x, t) plane, along which the solutions to the partial differential equation are constant. The characteristic curves for the wave equation are straight lines, which can be obtained by guessing a solution of the form

$$h(x,t) = f(x + \alpha t)\,, \tag{3.3.34}$$

for which

$$\frac{\partial^2 h}{\partial x^2} = f'', \qquad \text{and} \qquad \frac{\partial^2 h}{\partial t^2} = \alpha^2 f''\,. \tag{3.3.35}$$

Clearly the wave equation is satisfied if $\alpha^2 = v^2$, or for $\alpha = \pm vt$.

A solution of the form $h(x, t) = f(x - vt)$ describes a shape that is simply translated in time to the right (along $+x$) with velocity v, while $h(x, t) = f(x+vt)$ translates the function in the opposite direction. Any function of the form $h(x - vt, x + vt)$, such as a superposition of left and right moving parts, is also a solution. It is easy to check that any stationary wave can be obtained from the superposition of oppositely travelling waves, since

$$\sin(kx - \omega t) + \sin(kx + \omega t) = 2\sin(kx)\cos(\omega t)\,. \tag{3.3.36}$$