3.3 Fourier analysis

3.3.1 Fourier Series

The procedure for decomposing the initial condition as a sum of terms proportional to $\sin(n\pi x/L)$ is an example of *Fourier transformation.* In fact, one can similarly obtain Fourier series for any function defined on any interval. For example consider all functions $f(\theta)$ which are *periodic* in the interval 0 to 2π. The appropriate normal modes (basis functions) for such periodicity are $\cos(n\theta)$ for $n = 0, 1, 2, \cdots$, and $\sin(n\theta)$ for $n = 1, 2, \cdots$. We can then write for any periodic function

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] . \tag{3.3.1}$$

The coefficients in the expansion can be obtained using the following orthogonality relations

$$\begin{cases} \int_0^{2\pi} d\theta \cos(m\theta)\cos(n\theta) = \pi\delta_{mn} \\ \int_0^{2\pi} d\theta \sin(m\theta)\sin(n\theta) = \pi\delta_{mn} \\ \int_0^{2\pi} d\theta \sin(m\theta)\cos(n\theta) = 0 \end{cases} , \tag{3.3.2}$$

as

$$\begin{cases} A_0 = \frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta) \\ A_n = \frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta)\cos(n\theta) \\ B_n = \frac{1}{\pi}\int_0^{2\pi} d\theta f(\theta)\sin(n\theta) \end{cases} . \tag{3.3.3}$$

As a first example, consider a "square wave," i.e. a function

$$f_1(\theta) = \begin{cases} 1 & \text{for } 0 < \theta < \pi \\ -1 & \text{for } \pi < \theta < 2\pi \end{cases} . \tag{3.3.4}$$

The coefficients of the Fourier series are

$$A_n = \frac{1}{\pi}\left[\int_0^{\pi} d\theta \cos(n\theta) - \int_{\pi}^{2\pi} d\theta \cos(n\theta)\right] = 0 \,, \tag{3.3.5}$$

and

$$B_n = \frac{1}{\pi}\left[\int_0^{\pi} d\theta \sin(n\theta) - \int_{\pi}^{2\pi} d\theta \sin(n\theta)\right] = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] . \tag{3.3.6}$$

Hence we can construct a square wave from

$$f_1(\theta) = \frac{4}{\pi}\left[\sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \cdots\right] . \tag{3.3.7}$$

The square wave is certainly not continuous at the boundaries of the interval $[0, 2\pi]$, but the Fourier decomposition still works. Note that using periodicity of 2π, $f_1(-\theta) = -f_1(\theta)$ is an odd function of angle, and hence only the sine terms contribute to the Fourier series.

The procedure can be extended to any interval $x \in [a, b]$. As a second example, consider

$$f_2(x) = x, \qquad \text{for } -L/2 < x < +L/2 . \tag{3.3.8}$$

The appropriate functions for this interval are $\cos(2n\pi x/L)$ and $\sin(2n\pi x/L)$. Since the function $f_2(x)$ is *odd*, i.e. $f_2(-x) = -f_2(x)$, all the coefficients of the even functions, $\cos(2n\pi x/L)$, are zero. The coefficient of $\sin(2n\pi x/L)$ is given by

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} dx x \sin\left(\frac{2n\pi x}{L}\right) = \frac{1}{n\pi}\left[-x\cos\left(\frac{2n\pi x}{L}\right) + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right|_{-L/2}^{L/2} \tag{3.3.9}$$

$$= -\frac{L}{n\pi}\cos(n\pi) . \tag{3.3.10}$$

The Fourier series for this function is thus

$$f_2(x) = \frac{L}{\pi}\left[\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{8\pi x}{L}\right) + \cdots\right] , \tag{3.3.11}$$

and differs from that of the square wave only by including the even harmonics with opposite sign!

3.3.2 Complex Exponentials

We can write the Fourier series in more compact form by using complex exponentials, as follows

$$f(\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n\frac{e^{in\theta} + e^{-in\theta}}{2} + B_n\frac{e^{in\theta} - e^{-in\theta}}{2i}\right] \equiv \sum_{-\infty}^{\infty} e^{in\theta}\tilde{f}_n . \tag{3.3.12}$$

Using our previous results for $\{A_n\}$ and $\{B_n\}$, we obtain $(n \geq 0)$

$$\tilde{f}_n = \frac{1}{2}(A_n - iB_n) = \frac{1}{2\pi}\int_0^{2\pi} d\theta f(\theta)(\cos(n\theta) - i\sin(n\theta)) = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{-in\theta}f(\theta) . \tag{3.3.13}$$

(It is easy to check that the above form also gives the correct result for $\tilde{f}_0 \equiv A_0$, and for $\tilde{f}_n = A_n + iB_n$.)[3] The Fourier components $\tilde{f}_n$ can also be obtained directly from the *orthogonality conditions*

$$\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(m-n)\theta} = \delta_{mn} , \tag{3.3.14}$$

where δ_{mn} is the Kronecker delta function introduced earlier.

[3] There is no double counting: in considering sine and cosine modes only positive integers are included, while both positive and negative integers are allowed for $\tilde{f}_n = \tilde{f}^*_{-n}$.

3.3.3 Fourier Integrals

It is even more useful to construct Fourier transformations for functions defined on an infinite line. The transformation, rather than being in terms of a discrete (albeit infinite) series, takes the form of an integral,

$$f(x) = \int_{-\infty}^{\infty} dx e^{ikx} \tilde{f}(k) \,, \tag{3.3.15}$$

where the *Fourier transform* of the function is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} f(x) \,. \tag{3.3.16}$$

The key to this transformation is the *orthogonality relation*

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x - x') \,, \tag{3.3.17}$$

where $\delta(x-a)$ known as the *Dirac delta-function* is a limiting function highly peaked at $x = a$, and zero every where else, such that

$$\int dx \delta(x-a) g(x) = g(a) \,, \tag{3.3.18}$$

for any function $g(x)$. Comparison of Eqs. (3.3.14) and (3.3.17) indicates the close correspondence between the discrete (Kronecker) and continuous (Dirac) delta-functions. However, whereas δ_{mn} is dimensionless, $\delta(x-a)$ is a density and carries inverse dimensions of x. Thinking of the continuous delta-function as a density helps to understand why its value is infinite when its argument is zero: a unit value has to be obtained by integrating this density over an infinitesimal interval.

3.3.4 Diffusion

Consider density $n(x,t)$ that evolves according to the diffusion equation,

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} \,, \tag{3.3.19}$$

starting with an initial condition which is entirely concentrated at the origin $x = 0$, as represented by $n(x,0) = N\delta(x)$. The Dirac delta function is quite frequently used as a *source* in linear PDEs. The evolution of density in time is obtained as follows:

- The initial condition is decomposed in terms of its Fourier components; from Eq. (3.3.16), we find

$$\tilde{n}(k,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} n(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} N\delta(x) = \frac{N}{2\pi} \,. \tag{3.3.20}$$

- Noting that the general separable solution had time dependence $e^{i\omega t}$, from Eqs. (3.2.7) and (3.2.8), and the dispersion relation $i\omega(k) = -Dk^2$, each Fourier component evolves in time as

$$\tilde{n}(k,t) = e^{i\omega(k)t}\tilde{n}(k,0) = e^{-Dk^2t}\tilde{n}(k,0) = \frac{N}{2\pi}e^{-Dk^2t} . \tag{3.3.21}$$

- The solution in x basis at time t is obtained by inverting the Fourier transformation, an in Eq.(3.3.16), resulting in

$$n(x,t) = \int_{-\infty}^{\infty} dx\ e^{ikx}\tilde{n}(k.t) = \frac{N}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikx-Dk^2t} . \tag{3.3.22}$$

- Finally, using the Fourier transform of a *Gaussian* function (see below), we arrive at

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right) . \tag{3.3.23}$$

The initial density does spreads over time into a bell-shaped (Gaussian) form centered at the origin, but distributed over a distance that grows as $\sqrt{Dt}$. At the same time, the peak heigh of density decreases as $1/\sqrt{Dt}$, and the integral over the x remains at the initial value of N. It is also interesting to gain another perspective on the Dirac delta-function as the $t \to 0$ limit of a normalized Gaussian of width proportional to t.

At the very beginning of this course, we encountered the exponential function as the solution to the ODE $f'(x) = -\gamma f(x)$. It is interesting to note that all the explicit solutions that we have presented since have involved the exponential in one form or another, e.g. as in sums forming trigonometric functions.The Gaussian in Eq. (3.3.23) is the first truly independent form that we have seen, coming as a solution to the diffusion equation. It is very important to remember the following properties of Gaussian functions:

- A normalized Gaussian function,

$$G(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] ; \tag{3.3.24}$$

  a bell shaped curve peaked at $x = x_0$ that spreads symmetrically around x_0 over a characteristic width of order σ.

- The included prefactor ensures that $\int_{-\infty}^{\infty} dx\ G(x) = 1$, and that it can thus be regarded as a probability density. Using this weight, we find

$$\int_{-\infty}^{\infty} dx\ x\ G(x) = x_0 , \qquad \text{and} \qquad \int_{-\infty}^{\infty} dx\ (x-x_0)^2\ G(x) = \sigma^2 . \tag{3.3.25}$$

- The Fourier transform of a Gaussian is a similar (albeit not normalized) Gaussian

$$\int_{-\infty}^{\infty} dx\ e^{ikx}\ \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right] = \exp\left[-ikx_0 - \frac{\sigma^2k^2}{2}\right] . \tag{3.3.26}$$

3.3.5 Wave packet

Let us next use Fourier integrals to obtain the solution to the wave equation

$$\frac{\partial^2 h}{\partial t^2} = v^2 \frac{\partial^2 h}{\partial x^2}\,, \tag{3.3.27}$$

starting from an initial *stationary* Gaussian shape

$$h(x, t = 0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \text{with} \qquad \dot{h}(x, t = 0) = 0\,. \tag{3.3.28}$$

We can decompose this initial shape into a Fourier integral

$$h(x, 0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{h}(k, 0)\,, \tag{3.3.29}$$

which we the use of Eq. (3.3.26) yields

$$\tilde{h}(k, 0) = \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-ikx - \frac{x^2}{2\sigma^2}\right) = \exp\left(-\frac{\sigma^2 k^2}{2}\right)\,. \tag{3.3.30}$$

The dispersion relation for the wave equation $\omega^2 = v^2 k^2$ has two branches with $\omega = \pm vk$. For mode of wavenumber k, time evolution of the form

$$\tilde{h}(k, t) = \tilde{h}(k, 0) \left[f(k) e^{ikvt} + (1 - f(k)) e^{-ikvt}\right]\,, \tag{3.3.31}$$

is a valid solution for any $f(k)$. The choice of $f(k)$ is dictated by the initial condition. In this case $\dot{h}(x, t = 0) = 0$, requires $\dot{\tilde{h}}(k, t = 0) = 0$, i.e.

$$\dot{\tilde{h}}(k, t) = \tilde{h}(k, 0) \left[ikvf(k) e^{ikvt} - ikv(1 - f(k)) e^{-ikvt}\right] = 0 \quad \Rightarrow \quad f(k) = \frac{1}{2}\,. \tag{3.3.32}$$

This leads to

$$\begin{aligned} h(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) e^{ikx} \cos(vkt) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) e^{ikx} \frac{e^{ikvt} + e^{-ikvt}}{2} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{h}(k, 0) \left(e^{ik(x+vt)} + e^{ik(x-vt)}\right) \\ &= \frac{1}{2} \left[h(x + vt, 0) + h(x - vt, 0)\right]\,. \end{aligned} \tag{3.3.33}$$

This solution describes two Gaussian bumps, each half in size, one moving to the left, and the other to the right with velocity v. As long as these bumps are far away from any boundaries they maintain their shapes. Each bump represents a *traveling wave*, a disturbance that propagates without changing its shape.

The above example indicates that the wave equation admits solutions very different from the separable solutions that are also known as *standing waves*. There is in fact another procedure for solving partial differential equations, known as *the method of characteristics*, which naturally yields travelling wave solutions. In this method one searches for *characteristic curves* in the (x, t) plane, along which the solutions to the partial differential equation are constant. The characteristic curves for the wave equation are straight lines, which can be obtained by guessing a solution of the form

$$h(x,t) = f(x + \alpha t)\,, \tag{3.3.34}$$

for which

$$\frac{\partial^2 h}{\partial x^2} = f'', \qquad \text{and} \qquad \frac{\partial^2 h}{\partial t^2} = \alpha^2 f''\,. \tag{3.3.35}$$

Clearly the wave equation is satisfied if $\alpha^2 = v^2$, or for $\alpha = \pm vt$.

A solution of the form $h(x,t) = f(x - vt)$ describes a shape that is simply translated in time to the right (along $+x$) with velocity v, while $h(x,t) = f(x+vt)$ translates the function in the opposite direction. Any function of the form $h(x-vt, x+vt)$, such as a superposition of left and right moving parts, is also a solution. It is easy to check that any stationary wave can be obtained from the superposition of oppositely travelling waves, since

$$\sin(kx - \omega t) + \sin(kx + \omega t) = 2\sin(kx)\cos(\omega t)\,. \tag{3.3.36}$$

Recap

- On a finite interval of size L, $\sin(2n\pi x/L)$ and $\cos(2n\pi x/L)$ for $n = 0, 1, 2, \cdots$ form a complete basis, in terms of which any function $f(x)$ on the interval can be decomposed in a Fourier series.

- Along the infinite line $\sin(kx)$ and $\cos(kx)$ (for $k \geq 0$), or e^{ikx} (for real k) acts as a basis, for Fourier decomposition:

$$f(x) = \int_{-\infty}^{\infty} dx e^{ikx} \tilde{f}(k)\,, \quad \text{and} \quad \tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} f(x)\,. \tag{3.3.37}$$

- The orthogonality condition of Fourier modes is expressed in terms of the Dirac delta-function, as

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x - x')\,, \tag{3.3.38}$$