3.4 Scalar fields in higher dimensions

3.4.1 Locality, uniformity, and isotropy

In the previous sections we considered a field $u(x)$ defined along a one dimensional interval. There are many more cases in which we are interested in the variations of a field in higher dimensions. Two dimensional examples include waves on the surface of water, vibrations of a drum or a soap film, distortions of a membrane or plate. The temperature or pressure of a gas are instances from three dimensions, all corresponding to a scalar (single component) field. (The velocity field of a fluid, electric and magnetic fields, are examples of vector fields that will not be discussed in this section.)

A particular configuration of the field in d-dimensional space shall be represented by the function $h(\mathbf{x})$, where the vector $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ indicates a position in this space. As in Eq. (3.1.10) we would like to construct a force density $\mathcal{F}$, which is a functional of the configuration $h(\mathbf{x})$. We shall proceed by making the following assumptions:

- *Locality:* We assume that the force density at a particular location $\mathbf{x}$ is a reasonably smooth functional that depends on the value of the function at $\mathbf{x}$ and "nearby" points. What is meant by the latter is the relevant variations of are over sufficiently long scales (compared to any underlying microscopic physical scale) that they can be adequately be captured by the first few terms of a gradient expansion. (Much like the first few terms of a Taylor series present a good enough description of a smooth function for small amplitudes.) Since $\mathcal{F}$ is a scalar, using the summation convention, we generalize Eq. (3.1.10) to

$$\mathcal{F}(\mathbf{x}) = A_0 + A_1 h + B_i \partial_i h + C_{ij} \partial_i \partial_j h + \cdots + A_2 \frac{h^2}{2} + D_i \, h \partial_i h + E_{ij} \, \partial_i h \partial_j h + \cdots . \quad (3.4.1)$$

  Here, $(A_0 . A_1, A_2)$ are familiar coefficients in a Taylor expansion of a function, $\{B_i\}$ indicate components of a d-dimensional vector that appear as coefficients of gradient terms ($\partial_i h = \frac{\partial h}{\partial x_i}$). The coefficients of the set of second derivatives $\partial_i \partial_j h = \frac{\partial^2 h}{\partial x_i \partial x_j}$ from a $d \times d$ matrix $\mathbf{C}$. We can construct higher order terms in the series using various powers of h and combinations of partial derivatives, using the rule that any vector index must appear twice in accordance with the summation convention.

- *Uniformity:* In principle all coefficients appearing in Eq. (3.4.1) could vary as a function of position $\mathbf{x}$. This was also a possibility for the one dimensional force density Eq. (3.4.1), and even earlier in Eq. (3.1.6) for the force on an elastic band. Its absence in the latter case was due to the assumption that all the underlying springs in Eq. (3.1.1) are identical. Similarly, for a *uniform* system in which all positions in space are equivalent, the coefficients appearing in Eq. (3.4.1) will be constants independent of $\mathbf{x}$.

- *Isotropy:* The coefficients $\{B_i\}$ and $\{D_i\}$ thus represent vectors pointing to particular directions in $\mathbf{x}$ space that are intrinsic to the problem under study. Now consider a featureless space such as the surface of water at rest in a bucket. Ignoring the edges of the bucket, all directions along the two dimensional surface are equivalent. Since there is no intrinsic direction in this problem (along the surface), the coefficients $\{B_i\}$ and $\{D_i\}$ must be zero in this case.[4] The equivalence of all directions, known as *isotropy* constrains all terms in the expansion. For example, the only possible isotropic matrices are proportional to unity, i.e. $C_{ij} \propto D_{ij} \propto \delta_{ij}$. Thus, for a uniform and isotropic system Eq. (3.4.1) simplifies to

$$\mathcal{F}(\mathbf{x}) = A_0 + A_1 h + C\nabla^2 h + \cdots + A_2 \frac{h^2}{2} + E(\nabla h)^2 + \cdots . \tag{3.4.2}$$

- *Stability:* If the configuration $h(\mathbf{x}) = 0$ represents stable equilibrium, we are constrained as before to set $A_0 = 0$, $A_1 < 0$, and $C > 0$. We thus end up with a simple generalization of Eq. (3.1.11) to

$$\mathcal{F}(\mathbf{x}) = -Jh + K\nabla^2 h\,. \tag{3.4.3}$$

- Once more, with an expansion limited to the two terms in Eq. (3.4.3), the force density can be obtained from gradient descent in a functional

$$V[h(\mathbf{x})] = \int d^d x \left[\frac{J}{2} h^2 + \frac{K}{2} (\nabla h)^2\right] . \tag{3.4.4}$$

[4] If we instead consider water flowing along a pipe, the local flow velocity $\vec{v}$ provides a particular direction, and the force density does admit $\vec{B} \propto \vec{v}$, describing *advection*.