3.4.2 Can you hear the shape of a drum?

As for a one dimensional field, we can argue that the force density leads to a generic equation of motion of the form

$$\eta\frac{\partial h}{\partial t}+\rho\frac{\partial^2 h}{\partial t^2}=\mathcal{F}(x)\approx -Jh+K\nabla^2 h+\cdots\,. \tag{3.4.5}$$

The next time you encounter such a PDE in a novel equation, as you surely will in myriad contexts, you should not be surprised. Rather, if you see any other combination of derivatives you should inquire as to which of the very general arguments presented before have been violated.

We can again find solution to this higher dimensional PDE that have the separable form

$$h(\mathbf{x},t)=H(\mathbf{x})T(t)\,. \tag{3.4.6}$$

Substituting this form in Eq. (3.4.5), and dividing by $h = HT$, leads to

$$\frac{\rho\ddot{T}+\eta\dot{T}+JT}{T}=K\frac{\nabla^2 H}{H}\,. \tag{3.4.7}$$

As the left hand side is only a function of time t, while the right hand side only depends on $\mathbf{x}$, both sides must be constants. Setting the constant to λ, results in

$$\rho\ddot{T}+\eta\dot{T}+JT=\lambda T\,, \tag{3.4.8}$$

as before for the temporal component. While the spatial part leads to the d-dimensional PDE

$$K\nabla^2 H=\lambda H\,. \tag{3.4.9}$$

Solving the problem thus comes down to finding the "normal modes" encoded in Eq. (3.4.9). This is much more complicated than in one dimension due to boundary conditions. In one dimension by considering the two edges of allowed interval we could find (quantized) modes related to e^{ikx}. The boundary of a d-dimensional domain, however, can be a complicated $(d-1)$ dimensional manifold, and finding corresponding modes is no easy task.

A simple physical realization of the two-dimensional version of Eq. (3.4.5) is provided by a soap film. Consider a soap film which in equilibrium rests flat on a planar frame. The film is pushed out of equilibrium by blowing on it, and the distorted shape is described by its height $h(x,y,t)$. What is the subsequent motion of the film? In the same way that a string or rubber band under tension minimizes its length subject to boundary conditions, a soap film on a frame minimizes its area. The driving force behind this process is called the *surface tension* S, and the (potential) energy of a film of area A is simply $V = SA$. The element of area for the distorted film is $dA = dx\ dy\ \sqrt{1+(\nabla h)^2}$, resulting in a surface tension energy

$$V[h]=S\int d^2x\sqrt{1+(\nabla h)^2}=S\int d^2x\left[1+\frac{1}{2}(\nabla h)^2+\cdots\right]\,, \tag{3.4.10}$$

leading to a force density[5]

$$\mathcal{F}(\mathbf{x}) = -\frac{\delta V}{\delta h(\mathbf{x})} = S\frac{\nabla^2 h}{\sqrt{1+(\nabla h)^2}} \approx S\nabla^2 h\,, \tag{3.4.12}$$

to the lowest order. Note that a term of the form $-Jh$ is absent since the choice of $h = 0$ is arbitrary.

Assuming a mass density of ρ per unit area of the film as well as negligible friction ($\eta = 0$), we arrive at the two dimensional wave equation,

$$\rho\frac{\partial^2 h(x,y,t)}{\partial t^2} = S\nabla^2 h(x,y,t), \quad \Rightarrow \quad \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \nabla^2 h\,, \tag{3.4.13}$$

where $v = \sqrt{S/\rho}$ is the appropriate wave velocity. To find the normal modes oscillating with frequency ω_n we have to find solutions to $\nabla^2 H = -(\omega_n/v)^2 H$ that vanish at the boundaries of the film.

In a famous article, the mathematician *Mark Kac* asked the inverse question "Can one hear the shape of a drum?", i.e. given a set of normal mode frequencies ω_n captured from beating a drum, can one deduce its shape. Tackling this interesting question is far beyond the scope of our material; instead we shall take on the more modest task of distinguishing between the modes of rectangular and circular drums.

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[5]Following the general rule of functional integrals, if

$$V[h] = \int dx\; dy\; f(h, \partial_x h, \partial_y h)\,, \quad \text{then} \quad \frac{\delta V}{\delta h(x,y)} = \frac{\partial f}{\partial h} - \frac{\partial}{\partial x}\frac{\partial f}{\partial_x h} - \frac{\partial}{\partial y}\frac{\partial f}{\partial_y h}\,. \tag{3.4.11}$$