3.4.3 Normal modes with a rectangular frame

Consider a drum head, or a soap film, stretched on a wire frame. The deformations about the flat shape must vanish on the frame, similar to the (Dirichlet) pinned ends of a rubber band. For a rectangular domain, extending in the x direction from 0 to L_x, and in the y direction from 0 to L_y, it is most natural to search for a separable solution of the form

$$h(x,y,t) = X(x)Y(y)T(t)\,. \tag{3.4.14}$$

Substituting this form into Eq. (3.4.13), and dividing by $h = XYT$ leads to

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\,. \tag{3.4.15}$$

Since each term in the above equation depends on a separate argument, the only way for the equality to hold is if each term is independent of its argument, i.e. a constant. In analogy to the one dimensional case, we define the constants as

$$\frac{T''(t)}{T(t)} = -\omega^2, \qquad \frac{X''(x)}{X(x)} = -k_x^2, \qquad \frac{Y''(y)}{Y(y)} = -k_y^2\,, \tag{3.4.16}$$

where ω is the angular frequency, and $\mathbf{k} = (k_x, k_y)$ is the two dimensional *wave-vector*. The frequency and wave-vector are related by the *dispersion relation*

$$\omega^2 = v^2\left(k_x^2 + k_y^2\right), \quad \Rightarrow \quad \omega = vk\,, \tag{3.4.17}$$

where $k \equiv \sqrt{k_x^2 + k_y^2}$ is the *magnitude* of the wave-vector.

The problem is now reduced to three SHO equations for the functions T, X, and Y, and hence admits the general solution

$$h(x,y,t) = A\sin\left(k_x x + \theta_x\right)\sin\left(k_y y + \theta_y\right)\cos\left(\omega t + \phi\right)\,. \tag{3.4.18}$$

To satisfy the boundary conditions of vanishing h at $x = 0$ and $y = 0$, we must set $\theta_x = \theta_y = 0$, while the boundary conditions at the other two edges constrain $k_x L_x$ and $k_y L_y$ to be multiples of π. Hence the normal modes of the soap film on a rectangular frame are given by

$$h_{mn}(x,y,t) = A_{mn}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\cos\left(\omega_{mn} t + \phi\right)\,, \tag{3.4.19}$$

for integer n and m. The corresponding wave-number and frequency are

$$\mathbf{k}_{mn} = \pi\left(\frac{m}{L_x}, \frac{n}{L_y}\right) \quad \Rightarrow \quad \omega_{mn} = v\pi\sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}}\,. \tag{3.4.20}$$