3.4.4 Radial Laplacian on a circle

In many two dimensional situations the natural boundary conditions and corresponding deformations have circular, rather than rectangular symmetry. An example is provided by a soap film on a circular frame of radius R. Let us initially focus on deformations that are independent of the polar angle θ and only depend on a radial distance r from a central point, such that $h(x, y, t) = h(r, t)$. To calculate the forces appropriate to this case consider a infinitesimal ring on the surface, from r to $r + dr$. Since $h(r)$ is a deformation out of the plane at this point, the length of a segment stretching radially on the field from r to $r + dr$ satisfies $d\ell^2 = dr^2 + dh^2$, i.e. $d\ell = dr\sqrt{1 + (dh/dr)^2}$. The area of the ring is thus extended by the deformation to $2\pi r d\ell = 2\pi r dr\sqrt{1 + (dh/dr)^2}$, resulting in a surface tension energy (see Eq. (3.4.10))

$$V[h] = S\int_0^R (2\pi r dr)\sqrt{1 + \left(\frac{\partial h}{\partial r}\right)^2}\,, \tag{3.4.21}$$

leading to a force on the ring

$$\mathcal{F}(r) = -\frac{\delta V}{\delta h(r)} = S\frac{\partial}{\partial r}\left[2\pi r\frac{\frac{\partial h}{\partial r}}{\sqrt{1 + \left(\frac{\partial h}{\partial r}\right)^2}}\right] \approx S\frac{\partial}{\partial r}\left(2\pi r\frac{\partial h}{\partial r}\right)\,. \tag{3.4.22}$$

This “force density” actually acts on an infinitesimal ring of mass $(2\pi r dr)\rho$ (where ρ is the mass density), resulting in the equation of motion

$$\rho\,(2\pi r dr)\,\frac{\partial^2 h(r,t)}{\partial t^2} = S dr\frac{\partial}{\partial r}\left(2\pi r\frac{\partial h}{\partial r}\right)\,. \tag{3.4.23}$$

Dividing by $2\pi r S$, and setting $v^2 = S/\rho$ gives

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\,. \tag{3.4.24}$$

Note that despite the fact that $h(r)$ depends on only one radial coordinate, the form of $\nabla^2 h$ is different from that of a simple second derivative in the one dimensional case.