3.4 Scalar fields in higher dimensions

3.4.1 Locality, uniformity, and isotropy

In the previous sections we considered a field $u(x)$ defined along a one dimensional interval. There are many more cases in which we are interested in the variations of a field in higher dimensions. Two dimensional examples include waves on the surface of water, vibrations of a drum or a soap film, distortions of a membrane or plate. The temperature or pressure of a gas are instances from three dimensions, all corresponding to a scalar (single component) field. (The velocity field of a fluid, electric and magnetic fields, are examples of vector fields that will not be discussed in this section.)

A particular configuration of the field in d-dimensional space shall be represented by the function $h(\mathbf{x})$, where the vector $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ indicates a position in this space. As in Eq. (3.1.10) we would like to construct a force density $\mathcal{F}$, which is a functional of the configuration $h(\mathbf{x})$. We shall proceed by making the following assumptions:

- *Locality:* We assume that the force density at a particular location $\mathbf{x}$ is a reasonably smooth functional that depends on the value of the function at $\mathbf{x}$ and "nearby" points. What is meant by the latter is the relevant variations of are over sufficiently long scales (compared to any underlying microscopic physical scale) that they can be adequately be captured by the first few terms of a gradient expansion. (Much like the first few terms of a Taylor series present a good enough description of a smooth function for small amplitudes.) Since $\mathcal{F}$ is a scalar, using the summation convention, we generalize Eq. (3.1.10) to

$$\mathcal{F}(\mathbf{x}) = A_0 + A_1 h + B_i \partial_i h + C_{ij} \partial_i \partial_j h + \cdots + A_2 \frac{h^2}{2} + D_i\ h \partial_i h + E_{ij}\ \partial_i h \partial_j h + \cdots . \quad (3.4.1)$$

  Here, $(A_0 . A_1, A_2)$ are familiar coefficients in a Taylor expansion of a function, $\{B_i\}$ indicate components of a d-dimensional vector that appear as coefficients of gradient terms ($\partial_i h = \frac{\partial h}{\partial x_i}$). The coefficients of the set of second derivatives $\partial_i \partial_j h = \frac{\partial^2 h}{\partial x_i \partial x_j}$ from a $d \times d$ matrix $\mathbf{C}$. We can construct higher order terms in the series using various powers of h and combinations of partial derivatives, using the rule that any vector index must appear twice in accordance with the summation convention.

- *Uniformity:* In principle all coefficients appearing in Eq. (3.4.1) could vary as a function of position $\mathbf{x}$. This was also a possibility for the one dimensional force density Eq. (3.4.1), and even earlier in Eq. (3.1.6) for the force on an elastic band. Its absence in the latter case was due to the assumption that all the underlying springs in Eq. (3.1.1) are identical. Similarly, for a *uniform* system in which all positions in space are equivalent, the coefficients appearing in Eq. (3.4.1) will be constants independent of $\mathbf{x}$.

- *Isotropy:* The coefficients $\{B_i\}$ and $\{D_i\}$ thus represent vectors pointing to particular directions in $\mathbf{x}$ space that are intrinsic to the problem under study. Now consider a featureless space such as the surface of water at rest in a bucket. Ignoring the edges of the bucket, all directions along the two dimensional surface are equivalent. Since there is no intrinsic direction in this problem (along the surface), the coefficients $\{B_i\}$ and $\{D_i\}$ must be zero in this case.[4] The equivalence of all directions, known as *isotropy* constrains all terms in the expansion. For example, the only possible isotropic matrices are proportional to unity, i.e. $C_{ij} \propto D_{ij} \propto \delta_{ij}$. Thus, for a uniform and isotropic system Eq. (3.4.1) simplifies to

$$\mathcal{F}(\mathbf{x}) = A_0 + A_1 h + C\nabla^2 h + \cdots + A_2\frac{h^2}{2} + E(\nabla h)^2 + \cdots . \tag{3.4.2}$$

- *Stability:* If the configuration $h(\mathbf{x}) = 0$ represents stable equilibrium, we are constrained as before to set $A_0 = 0$, $A_1 < 0$, and $C > 0$. We thus end up with a simple generalization of Eq. (3.1.11) to

$$\mathcal{F}(\mathbf{x}) = -Jh + K\nabla^2 h\,. \tag{3.4.3}$$

- Once more, with an expansion limited to the two terms in Eq. (3.4.3), the force density can be obtained from gradient descent in a functional

$$V[h(\mathbf{x})] = \int d^d x \left[\frac{J}{2}h^2 + \frac{K}{2}(\nabla h)^2\right] . \tag{3.4.4}$$

3.4.2 Can you hear the shape of a drum?

As for a one dimensional field, we can argue that the force density leads to a generic equation of motion of the form

$$\eta\frac{\partial h}{\partial t} + \rho\frac{\partial^2 h}{\partial t^2} = \mathcal{F}(x) \approx -Jh + K\nabla^2 h + \cdots . \tag{3.4.5}$$

The next time you encounter such a PDE in a novel equation, as you surely will in myriad contexts, you should not be surprised. Rather, if you see any other combination of derivatives you should inquire as to which of the very general arguments presented before have been violated.

We can again find solution to this higher dimensional PDE that have the separable form

$$h(\mathbf{x}, t) = H(\mathbf{x})T(t)\,. \tag{3.4.6}$$

Substituting this form in Eq. (3.4.5), and dividing by $h = HT$, leads to

$$\frac{\rho\ddot{T} + \eta\dot{T} + JT}{T} = K\frac{\nabla^2 H}{H}\,. \tag{3.4.7}$$

[4] If we instead consider water flowing along a pipe, the local flow velocity $\vec{v}$ provides a particular direction, and the force density does admit $\vec{B} \propto \vec{v}$, describing *advection*.

As the left hand side is only a function of time t, while the right hand side only depends on $\mathbf{x}$, both sides must be constants. Setting the constant to λ, results in

$$\rho\ddot{T} + \eta\dot{T} + JT = \lambda T\,, \tag{3.4.8}$$

as before for the temporal component. While the spatial part leads to the d-dimensional PDE

$$K\nabla^2 H = \lambda H\,. \tag{3.4.9}$$

Solving the problem thus comes down to finding the "normal modes" encoded in Eq. (3.4.9). This is much more complicated than in one dimension due to boundary conditions. In one dimension by considering the two edges of allowed interval we could find (quantized) modes related to e^{ikx}. The boundary of a d-dimensional domain, however, can be a complicated $(d-1)$ dimensional manifold, and finding corresponding modes is no easy task.

A simple physical realization of the two-dimensional version of Eq. (3.4.5) is provided by a soap film. Consider a soap film which in equilibrium rests flat on a planar frame. The film is pushed out of equilibrium by blowing on it, and the distorted shape is described by its height $h(x, y, t)$. What is the subsequent motion of the film? In the same way that a string or rubber band under tension minimizes its length subject to boundary conditions, a soap film on a frame minimizes its area. The driving force behind this process is called the *surface tension* S, and the (potential) energy of a film of area A is simply $V = SA$. The element of area for the distorted film is $dA = dx\ dy\ \sqrt{1 + (\nabla h)^2}$, resulting in a surface tension energy

$$V[h] = S\int d^2x\sqrt{1+(\nabla h)^2} = S\int d^2x\left[1 + \frac{1}{2}(\nabla h)^2 + \cdots\right]\,, \tag{3.4.10}$$

leading to a force density[5]

$$\mathcal{F}(\mathbf{x}) = -\frac{\delta V}{\delta h(\mathbf{x})} = S\frac{\nabla^2 h}{\sqrt{1+(\nabla h)^2}} \approx S\nabla^2 h\,, \tag{3.4.12}$$

to the lowest order. Note that a term of the form $-Jh$ is absent since the choice of $h = 0$ is arbitrary.

Assuming a mass density of ρ per unit area of the film as well as negligible friction ($\eta = 0$), we arrive at the two dimensional wave equation,

$$\rho\frac{\partial^2 h(x,y,t)}{\partial t^2} = S\nabla^2 h(x,y,t), \quad \Rightarrow \quad \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \nabla^2 h\,, \tag{3.4.13}$$

where $v = \sqrt{S/\rho}$ is the appropriate wave velocity. To find the normal modes oscillating with frequency ω_n we have to find solutions to $\nabla^2 H = -(\omega_n/v)^2 H$ that vanish at the boundaries of the film.

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[5]Following the general rule of functional integrals, if

$$V[h] = \int dx\ dy\ f(h, \partial_x h, \partial_y h)\,, \quad \text{then} \quad \frac{\delta V}{\delta h(x,y)} = \frac{\partial f}{\partial h} - \frac{\partial}{\partial x}\frac{\partial f}{\partial_x h} - \frac{\partial}{\partial y}\frac{\partial f}{\partial_y h}\,. \tag{3.4.11}$$

In a famous article, the mathematician *Mark Kac* asked the inverse question "Can one hear the shape of a drum?", i.e. given a set of normal mode frequencies ω_n captured from beating a drum, can one deduce its shape. Tackling this interesting question is far beyond the scope of our material; instead we shall take on the more modest task of distinguishing between the modes of rectangular and circular drums.

3.4.3 Normal modes with a rectangular frame

Consider a drum head, or a soap film, stretched on a wire frame. The deformations about the flat shape must vanish on the frame, similar to the (Dirichlet) pinned ends of a rubber band. For a rectangular domain, extending in the x direction from 0 to L_x, and in the y direction from 0 to L_y, it is most natural to search for a separable solution of the form

$$h(x, y, t) = X(x)Y(y)T(t)\,. \tag{3.4.14}$$

Substituting this form into Eq. (3.4.13), and dividing by $h = XYT$ leads to

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\,. \tag{3.4.15}$$

Since each term in the above equation depends on a separate argument, the only way for the equality to hold is if each term is independent of its argument, i.e. a constant. In analogy to the one dimensional case, we define the constants as

$$\frac{T''(t)}{T(t)} = -\omega^2, \qquad \frac{X''(x)}{X(x)} = -k_x^2, \qquad \frac{Y''(y)}{Y(y)} = -k_y^2\,, \tag{3.4.16}$$

where ω is the angular frequency, and $\mathbf{k} = (k_x, k_y)$ is the two dimensional *wave-vector*. The frequency and wave-vector are related by the *dispersion relation*

$$\omega^2 = v^2\left(k_x^2 + k_y^2\right), \quad \Rightarrow \quad \omega = vk\,, \tag{3.4.17}$$

where $k \equiv \sqrt{k_x^2 + k_y^2}$ is the *magnitude* of the wave-vector.

The problem is now reduced to three SHO equations for the functions T, X, and Y, and hence admits the general solution

$$h(x, y, t) = A\sin\left(k_x x + \theta_x\right)\sin\left(k_y y + \theta_y\right)\cos\left(\omega t + \phi\right)\,. \tag{3.4.18}$$

To satisfy the boundary conditions of vanishing h at $x = 0$ and $y = 0$, we must set $\theta_x = \theta_y = 0$, while the boundary conditions at the other two edges constrain $k_x L_x$ and $k_y L_y$ to be multiples of π. Hence the normal modes of the soap film on a rectangular frame are given by

$$h_{mn}(x, y, t) = A_{mn}\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\cos\left(\omega_{mn}t + \phi\right)\,, \tag{3.4.19}$$

for integer n and m. The corresponding wave-number and frequency are

$$\mathbf{k}_{mn} = \pi\left(\frac{m}{L_x}, \frac{n}{L_y}\right) \quad \Rightarrow \quad \omega_{mn} = v\pi\sqrt{\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}}\,. \tag{3.4.20}$$

3.4.4 Circular symmetry

In many two dimensional situations the natural boundary conditions and corresponding deformations have circular, rather than rectangular symmetry. An example is provided by a soap film on a circular frame of radius R. Let us initially focus on deformations that are independent of the polar angle θ and only depend on a radial distance r from a central point, such that $h(x, y, t) = h(r, t)$. To calculate the forces appropriate to this case consider a infinitesimal ring on the surface, from r to $r + dr$. Since $h(r)$ is a deformation out of the plane at this point, the length of a segment stretching radially on the field from r to $r + dr$ satisfies $d\ell^2 = dr^2 + dh^2$, i.e. $d\ell = dr\sqrt{1 + (dh/dr)^2}$. The area of the ring is thus extended by the deformation to $2\pi r d\ell = 2\pi r dr\sqrt{1 + (dh/dr)^2}$, resulting in a surface tension energy (see Eq. (3.4.10))

$$V[h] = S\int_0^R (2\pi r dr)\sqrt{1 + \left(\frac{\partial h}{\partial r}\right)^2}\,, \tag{3.4.21}$$

leading to a force on the ring

$$\mathcal{F}(r) = -\frac{\delta V}{\delta h(r)} = S\frac{\partial}{\partial r}\left[2\pi r\frac{\frac{\partial h}{\partial r}}{\sqrt{1 + \left(\frac{\partial h}{\partial r}\right)^2}}\right] \approx S\frac{\partial}{\partial r}\left(2\pi r\frac{\partial h}{\partial r}\right)\,. \tag{3.4.22}$$

This "force density" actually acts on an infinitesimal ring of mass $(2\pi r dr)\rho$ (where ρ is the mass density), resulting in the equation of motion

$$\rho\,(2\pi r dr)\,\frac{\partial^2 h(r, t)}{\partial t^2} = S dr\frac{\partial}{\partial r}\left(2\pi r\frac{\partial h}{\partial r}\right)\,. \tag{3.4.23}$$

Dividing by $2\pi r S$, and setting $v^2 = S/\rho$ gives

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\,. \tag{3.4.24}$$

Note that despite the fact that $h(r)$ depends on only one radial coordinate, the form of $\nabla^2 h$ is different from that of a simple second derivative in the one dimensional case.

If we allow for variations in both the radial and polar directions, an infinitesimal element has size dr in the radial direction and $r\ d\phi$ in the tangential direction. The resulting area element is

$$dA = (dr)(r\ d\phi)\,. \tag{3.4.25}$$

By following variations of $h(r, \phi)$ along the sides of this element, we find

$$(\nabla h)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \left(\frac{\partial h}{r\partial\phi}\right)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial\phi}\right)^2\,. \tag{3.4.26}$$

The generalization of Eq. (3.4.24) now yields the equation of motion for the field $h(r\phi, t)$ on this area element as

$$\rho\,(dr\ rd\phi)\,\frac{\partial^2 h(r, \phi, t)}{\partial t^2} = S dr\ d\phi\left[\frac{\partial}{\partial r}\left(r\frac{\partial h(r, \phi, t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h(r, \phi, t)}{\partial\phi^2}\right]\,. \tag{3.4.27}$$

The wave equation in polar coordinates thus reads

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \nabla^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2} = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2} \,. \tag{3.4.28}$$

We can again try a separable solutions of the form

$$h(r,\phi,t) = f(r)\Phi(\phi)T(t) \,, \tag{3.4.29}$$

which after dividing the equation by $h = Tf\Phi$ yields

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{\Phi''}{\Phi} \,. \tag{3.4.30}$$

It is not immediately apparent that the Φ and f functions are independent. However, the angular dependence must satisfy the periodicity requirement $\Phi(\phi + 2\pi) = \Phi(\phi)$, and hence can be constructed as superposition of modes

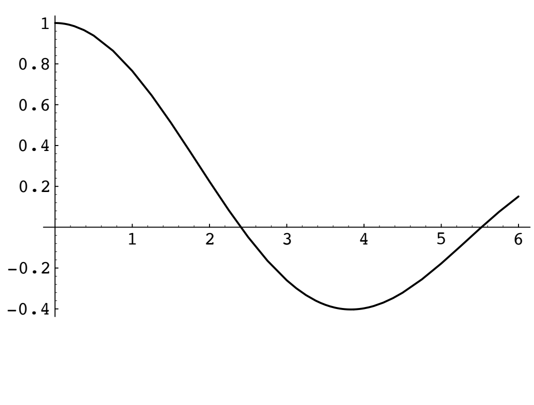
$$\Phi(\phi) \propto \cos(n\phi + \phi) \,, \quad \text{with} \quad \Phi'' = -n^2\Phi \quad \text{for integer } n \,. \tag{3.4.31}$$

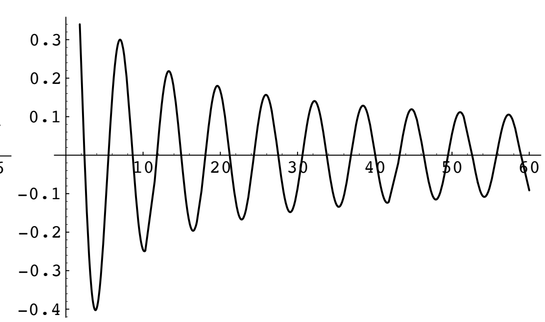
We can now set each side of Eq. (3.4.30) to a constant, say k^2. The left hand side then reduces to the standard SHO equation, with solutions $T(t) \propto \cos(kvt + \phi)$. The right hand side leads to an ordinary differential equation

$$f''(r) + \frac{1}{r}f'(r) - \frac{n^2}{r^2}f(r) = k^2 f(r) \,, \tag{3.4.32}$$

which we have not encountered before. The solution to this equation cannot be expressed simply in terms of algebraic or exponential functions. They are, however, rather well behaved function that have been tabulated, and are known as *Bessel functions*,

$$f(r) = AJ_n(kr) \quad \text{for integer } n \,. \tag{3.4.33}$$





The accompanying figure depicts the function J_0 at two scales. It resembles the cosine function in that it starts from unity with zero slope at the origin, and oscillates as a function of its argument. However, the zeros of the function are not equally spaced, and the magnitude of the function gets smaller for larger arguments.[6]

The zeros of $J_0(x)$ are important to finding the normal modes with closed boundary conditions. For a soap film on a circular ring of radius R, the vanishing of the deformation (and hence $f(R)$) at the edge requires $J_0(kR) = 0$. From the figure, we see that the first zero occurs for $k_1 R \approx 2.40$, and hence $k_1 \approx 2.40/R$, for a normal mode frequency of $\omega_1 \approx 2.40v/R$. The second zero occurs at 5.52, for $k_2 \approx 5.52/R$ and $\omega_2 \approx 5.52v/R$. The second mode has a line of zeros at a distance of $2.40/5.52 \approx 0.43$ towards the edge. Higher order modes have further nodal lines. Zeros of the Bessel function have been tabulated and can be used for calculating the corresponding normal mode frequencies.

By comparing the notes (frequencies) we can certainly distinguish between the rectangular and circular drums (and any other shape whose normal modes have been tabulated). Since each shape has in principle an infinite number of frequencies, it seems at first likely that distinct shapes should have distinct frequencies. However, in 1992 mathematicians constructed two different shapes with identical frequencies, thus providing a negative answer to Mark Kac's question "can one hear the shape of a drum?".

There are also analogs of *open* boundary conditions, such as a rigid circular plate forced to vibrate by oscillating its center. (This is known as the Chladni plate.) In this case the normal derivative, $\partial h/\partial r$, has to vanish at the boundary, requiring $J_0'(kR) = 0$. The zeros of this function have also been tabulated, with the first two occurring at 3.83 and 7.02. The tables demonstrate that the zeros and extrema at large orders become more regular with spacings that become closer and closer to π. Indeed, one can show that the Bessel function at large argument is given by the asymptotic formula

$$\lim_{x\to\infty} J_0(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) . \tag{3.4.34}$$

We can gain some physical understanding of some aspects of this formula by thinking about traveling wave solutions.

3.4.5 Planar and Circular Travelling Waves

One can also generate travelling waves on a soap film, or (more easily) on the surface of water, although the dispersion relation in the latter case is more complicated. The simplest type of solution varies only in one direction, e.g.

$$h(x, y, t) = A\cos(kx - \omega t + \theta) , \tag{3.4.35}$$

and is effectively one dimensional. These solutions also exist in three dimension and are called *plane waves*.

[6]As any second order differential equation there are two independent solutions. However, the other solution to this equation goes to infinity at $r = 0$, and does not arise in situations of interest to us.

More interesting are traveling solutions with circular symmetry. It is indeed easy to verify that

$$h(r,t) = AJ_0\left(kr - \omega t\right) , \tag{3.4.36}$$

is a solution to

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} , \tag{3.4.37}$$

provided that $\omega = kv$. This solution corresponds to a wave propagating out from the center. At small patch of this wave-from at large distances r should effectively look like a plane wave, hence $J_0(x)$ should be proportional to $\cos(x+\theta)$ at large x. The decay of the amplitude with r can be explained by noting that the input power at the origin, after travelling a distance r is uniformly distributed over a parameter of size $2\pi r$. Hence the local energy density must decay as $1/r$. Since the energy density is proportional to the square of the amplitude, the amplitude itself must decay as $1/\sqrt{r}$, as indicated in the asymptotic formula. This reasoning does not explain the phase factor of $\pi/4$, and the precise proportionality factor, which require matching to the solution at small r.

Recap

- Requirements of locality, uniformity and isotropy, lead to generic descriptions of long-wave length and small amplitude displacements that depend on the combination of second derivatives known as the Laplacian, which in Cartesian coordinates has the form

$$\nabla^2 h = \sum_{i=1}^{d} \frac{\partial^2 h}{\partial {x_i}^2} \equiv \partial_i \partial_i h \text{ using the summation convention.} \tag{3.4.38}$$

- In two dimensional polar coordinates (r, θ), the Laplacian takes the form

$$\nabla^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} . \tag{3.4.39}$$

- Normal modes of a circular drum involve Bessel functions.