3.5.1 Laplacian on a circle, including angular variations

Let us reexamine the results of Sec. 3.4.4 by allowing for variations in both the radial and polar directions. An infinitesimal element has size dr in the radial direction and $r\ d\phi$ in the tangential direction. The resulting area element is

$$dA = (dr)(r\ d\phi)\,. \tag{3.5.1}$$

By following variations of $h(r, \phi)$ along the sides of this element, we find

$$(\nabla h)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \left(\frac{\partial h}{r \partial \phi}\right)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \phi}\right)^2 . \tag{3.5.2}$$

The generalization of Eq. (3.4.24) now yields the equation of motion for the field $h(r\phi, t)$ on this area element as

$$\rho\,(dr\ rd\phi)\,\frac{\partial^2 h(r,\phi,t)}{\partial t^2} = S dr\ d\phi\left[\frac{\partial}{\partial r}\left(r\frac{\partial h(r,\phi,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h(r,\phi,t)}{\partial \phi^2}\right] . \tag{3.5.3}$$

The wave equation in polar coordinates thus reads

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \nabla^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2} = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2} . \tag{3.5.4}$$

We can again try separable solutions of the form

$$h(r, \phi, t) = f(r)\Phi(\phi)T(t)\,, \tag{3.5.5}$$

which after dividing the equation by $h = Tf\Phi$ yields

$$\frac{1}{v^2}\frac{T''(t)}{T(t)} = \frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{\Phi''}{\Phi} . \tag{3.5.6}$$

It is not immediately apparent that the Φ and f functions are independent. However, the angular dependence must satisfy the periodicity requirement $\Phi(\phi + 2\pi) = \Phi(\phi)$, and hence can be constructed as superposition of modes

$$\Phi(\phi) \propto \cos(n\phi + \theta_n)\,, \quad \text{with} \quad \Phi'' = -n^2\Phi \quad \text{for integer } n\,. \tag{3.5.7}$$

We can now set each side of Eq. (3.5.6) to a constant, say k^2. The left hand side then reduces to the standard SHO equation, with solutions $T(t) \propto \cos(kvt + \phi)$. The right hand side leads to the generalization of Eq. (3.4.27) to

$$f''(r) + \frac{1}{r}f'(r) - \frac{n^2}{r^2}f(r) = k^2 f(r)\,. \tag{3.5.8}$$

The solutions to this equation also cannot be expressed in terms of algebraic or exponential functions, and are known as n-th order *Bessel functions*,

$$f(r) = AJ_n(kr) \quad \text{for integer } n\,. \tag{3.5.9}$$