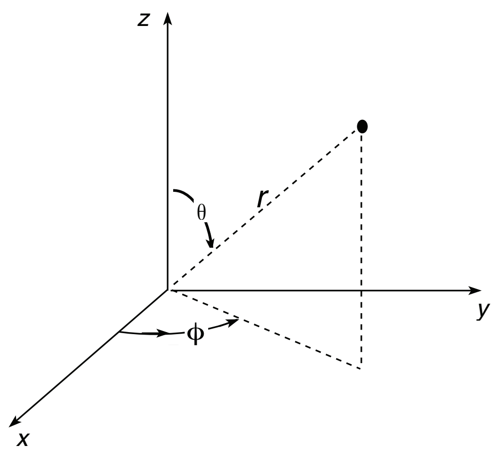
3.5.2 Spherical coordinates

In Sec. 3.4.4 we presented the form on the Laplacian operator, and its normal modes, in a system with circular symmetry. In addition to the radial coordinate r, a point is now indicated by two angles θ and ϕ, as indicated in the figure below. The original Cartesian coordinates are now related to the spherical coordinates by

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases}, \quad \text{for} \quad 0 \le \theta \le \pi \quad \text{and} \quad 0 \le \phi < 2\pi\,. \tag{3.5.10}$$



Infinitesimal variations in the radial and two angular directions, lead to changes dr, $r\ d\theta$, and $r\ \sin(\theta)\ d\phi$ respectively, resulting in the volume element

$$dV = (dr)(r\ d\theta)(r\ \sin(\theta)\ d\phi) = r^2\ \sin\theta\ dr\ d\theta\ d\phi\,. \tag{3.5.11}$$

Generalizing Eq. (3.5.2), by following variations of a scalar function $h(r,\theta,\phi)$ along the sides of this element, we find

$$(\nabla h)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \left(\frac{\partial h}{r\partial\theta}\right)^2 + \left(\frac{\partial h}{r\sin\theta\partial\phi}\right)^2 = \left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial\theta}\right)^2 \frac{1}{r^2\sin^2\theta}\left(\frac{\partial h}{\partial\phi}\right)^2 . \tag{3.5.12}$$

We can now find the Laplacian operator in spherical coordinates by considering the variations of

$$U = \frac{1}{2}\int dV(\nabla h)^2 = \frac{1}{2}\int [r^2\ \sin\theta\ dr\ d\theta\ d\phi]\left[\left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial h}{\partial\phi}\right)^2\right] . \tag{3.5.13}$$

As in Eq. (3.5.3), considerations of a "force density" leads to

$$[r^2 \ \sin\theta \ dr \ d\theta \ d\phi]\nabla^2 h = dr \ d\theta \ d\phi \left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial h}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial h}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 h}{\partial \phi^2}\right] . \tag{3.5.14}$$

The Laplacian in spherical coordinates is thus obtained as

$$\nabla^2 h = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial h}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial h}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 h}{\partial \phi^2} . \tag{3.5.15}$$