3.6.1 Continuity equation

We arrived at the form of the Laplacian operator in circular and spherical coordinates by considering variations of $U = \int dV(\nabla h)^2/2$. It is useful to gain a different perspective by examining the problem of diffusion of a density field $n(\vec{r}, t)$. We observed in Sec. (3.3.4) that the changes in density can be attributed to a current that moves the diffusing material around. While the density is a scalar field, the diffusive current is a vector field

$$\vec{J} = -D\vec{\nabla}n\,, \tag{3.6.1}$$

which smoothens density by moving diffusers from high to low concentrations. As already noted in Eq. (3.1.19), in one dimension

$$\frac{d}{dt}\int_a^b dxn(x,t) = D\int_a^b dx\frac{\partial^2 n}{\partial x^2} = D\left[\left.\frac{\partial n}{\partial x}\right|_b - \left.\frac{\partial u}{\partial x}\right|_a\right] = J(a) - J(b)\,, \tag{3.6.2}$$

the change in density for each interval is related to the difference between incoming and outgoing currents. Given the conservation of the net diffusive substance, we should be able to construct the generalization of the above equation to higher dimensions:

- Consider a region of d-dimensional space of volume V, bounded by a surface of area S. We expect the change in the amount of diffusive material within V to be related to the net flux of material taken in or out through the surface by the current $\vec{J}$. This can be written as

$$\frac{d}{dt}\int_{\text{volume } V} dV\ n = -\int_{\text{surface } S} \vec{J}\cdot d\vec{S}\,, \tag{3.6.3}$$

 where $d\vec{S}$ indicates the flux through a small element of surface of area dS; the direction of $d\vec{S}$ is taken to point outward, normal to the area element.

- For an infinitesimal volume element in the form of a hyper-cube, the above equality leads to

$$\dot{n}(\vec{r},t)\ dV = -\sum_{i=1}^{d} dS_i\left(\frac{\partial J_i}{\partial x_i}\ dx_i\right)\,. \tag{3.6.4}$$

 The contribution along each direction i comes from the difference in fluxes $\delta J_i = \frac{\partial J_i}{\partial x_i}dx_i$, passing through the orthogonal surfaces of area dS_i. For the hypercube, $dV = S_1\ dx_1 = S_2\ dx_2 = \cdots$, resulting in

$$\dot{n}(\vec{r},t) = -\sum_{i=1}^{d}\frac{\partial J_i}{\partial x_i}\,. \tag{3.6.5}$$

- For any vector field field $\vec{v}(\vec{r})$, a scalar *divergence* is defined by (using the summation convention)

$$\text{div}\ \vec{v} = \partial_i v_i = \vec{\nabla}\cdot\vec{v}\,. \tag{3.6.6}$$

- The conservation of (in this case diffusive) material can then be expressed by the *continuity equation*,

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \vec{J}\,, \qquad \frac{\partial n}{\partial t} = D\nabla^2 n\,, \qquad \text{for} \quad \vec{J} = -D\vec{\nabla} n\,. \tag{3.6.7}$$

- While the above formulae were motivated by conservation of matter during diffusion, Eq. (3.6.1) embodies the highly important and useful *divergence theorem*

$$\int_{\text{volume } V} dV \;\text{div}\vec{v} = -\int_{\text{surface } S} \vec{v} \cdot d\vec{S}\,, \tag{3.6.8}$$

  namely that the flux of a vector field $\vec{v}$ through any closed surface is equation to the integral of the divergence of $\vec{v}$ though the enclosed volume.