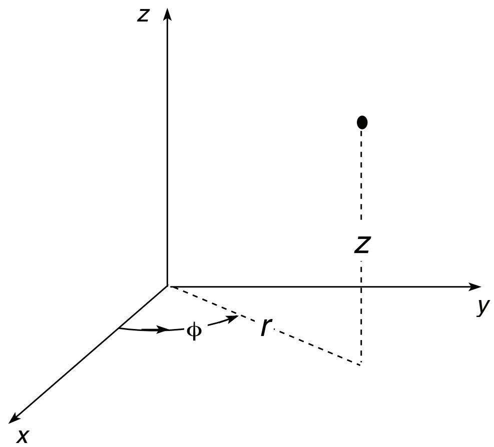
3.6.3 Cylindrical and spherical coordinates

Cylindrical coordinates simply extend the 2D polar coordinates, Eq. (3.6.11), by adding a third coordinate, z, pointing out of the plane.



From the Jacobian

$$\mathbf{J}(r,\phi,z) = \begin{bmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.6.26}$$

we obtain the metric factors

$$h_r = 1\,, \qquad h_\phi = r, \qquad h_z = 1\,, \quad \text{and} \quad |\det \mathbf{J}| = r\,. \tag{3.6.27}$$

For a scalar field $\Phi(r,\phi,z)$, we then obtain

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\phi}, \frac{\partial\Phi}{\partial z}\right), \tag{3.6.28}$$

and

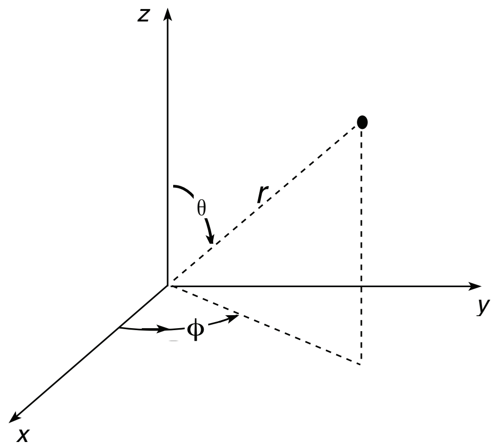
$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}\,. \tag{3.6.29}$$

A vector field can also be presented in these coordinates with components (v_r, v_ϕ, v_z) that are functions of (r,ϕ,z). The divergence of the vector is then obtained as

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}\,. \tag{3.6.30}$$

Spherical coordinates provide another representation of three dimensional space, replacing the axiz z with the angle θ to the z axis, such that $z = r\cos(\theta)$. With r now indicating the distance to the origin, its projection onto the 2D plane has length $r\sin(\theta)$, such that

$$x = r\sin(\theta)\cos(\phi) \quad y = r\sin(\theta)\sin(\phi) \quad z = r\cos(\theta)\,, \quad \text{with} \quad 0 \le \theta \le \pi \quad \text{and} 0 \le \phi < 2\pi\,. \tag{3.6.31}$$



The Jacobian associated with this transformation is

$$\mathbf{J}(r,\theta,\phi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{3.6.32}$$

From the magnitudes of the column vectors, we find

$$h_r = 1 \qquad h_\theta = r\,, \quad h_\phi = \sin\theta, \qquad \text{while} \quad |\det\mathbf{J}| = r^2\sin\theta\,. \tag{3.6.33}$$

For a scalar field $\Phi(r,\theta,\phi)$, we then have

$$\vec{\nabla}\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\,, \tag{3.6.34}$$

and

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\,. \tag{3.6.35}$$

For a vector field $\vec{v} = (v_r, v_\theta, v_\phi)$,

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,. \tag{3.6.36}$$