3.6.4 Dynamics of vector Fields

We introduced scalar fields in one dimension by considering the limit of a chain of N particles connected by springs or via a rubber band. While we assumed that the particles were constrained to move along the line, with displacements quantified by $\{u_n\}$, they could have well been allowed to move in two or three dimensions, with corresponding displacements indicated by a set of vectors $\{\vec{u}_n\}$. Taking the continuum limit ($na \to x \in [0, L = Na]$) now leads to a vector field $\vec{u}(x)$.

In the above example of the chain, the coordinate x is one dimensional, while the vector field $\vec{u}(x)$ can be 2, or 3 dimensional. More generally, we can consider vector fields $\vec{u}(\mathbf{x})$ where $\vec{u} = (u_1, u_2, \cdots, u_m)$ has m components spanning a d-dimensional space of coordinates $\mathbf{x} = (x_1, x_2, \cdots, x_d)$. In the majority of cases $m = d$, as for the case of the electric and magnetic fields in three dimensions. Another example, most directly generalizing displacements of a chain to three dimensions corresponds to the distortions $\vec{u}(\vec{x})$ of a solid in $m = d = 3$ dimensions. The gradient of a scalar field is also an example of a vector field.

Generalizing the approach we have followed for particles and scalar fields, we may seek to describe the changes in the vector field as a function of time, $\vec{u}(\mathbf{x}, t)$ via a vectorial variant of PDEs as

$$\eta \frac{\partial \vec{u}}{\partial t} + \rho \frac{\partial^2 \vec{u}}{\partial t^2} + \cdots = \vec{\mathcal{F}}(\mathbf{x}) \,, \tag{3.6.37}$$

where the vector force density depends on the vector field $\vec{u}(\mathbf{x})$ and its derivatives (in a gradient expansion) around the point $\mathbf{x}$.

Let us focus on the Taylor expansion of the force $\vec{\mathcal{F}}(\mathbf{x})$ for small distortions $\vec{u}(\mathbf{x})$, say in the relevant case of a solid or gel in three dimensions. We can followi the procedure described in Sec. 3.4.1, and employing the constrains of *locality* and *uniformity* construct a *gradient expansion* for the vectorial force field. The mathematical consequence of *isotropy*, however, is more interesting: In dealing with scalar quantities the contraction of two gradient operators lead to the term $\nabla^2 h$, for vectorial quantities, one gradient can be contracted with the vector field, resulting in a new contribution to the gradient expansion. In index notation, we thus find

$$\vec{\mathcal{F}}_\alpha(\mathbf{x}) = -J u_\alpha + K \partial_\beta \partial_\beta u_\alpha + L \partial_\alpha \partial_\beta u_\beta + \cdots \,; \tag{3.6.38}$$

in addition to the usual Laplacian– now for the vector field $\vec{u}(\vec{x})$, there is a contribution that is the gradient of the divergence of the field– $\nabla(\nabla \cdot \vec{u})$.

The dispersion relation governing the PDE of Eq. (3.6.37), with the force density in Eq.(3.6.38) is obtained by considering the (vectorial) trial solution

$$u_\alpha(\vec{x}) \propto \Re[e^{-i\omega t}] \Re[e^{i\vec{k}\cdot\vec{x}}] \hat{e}_\alpha \,, \tag{3.6.39}$$

with $\hat{e}_\alpha$ indicating the α-th component of a unit vector $\hat{e}$. Substituting the above form in the equation leads to

$$\left[-i\eta\omega - \rho\omega^2 + J\right] \hat{e}_\alpha = -\left[K k^2 \delta_{\alpha\beta} + L k_\alpha k_\beta\right] \hat{e}_\beta \equiv -\mathbf{K}_{\alpha\beta} \hat{e}_\beta \,, \tag{3.6.40}$$

with a matrix relation emerging from the mixed derivative $\partial_\alpha \partial_\beta$.

While the wave-vector $\vec{k}$ does introduce a particular direction, the matrix $\mathbf{K}$ still reflects our assumption of isotropy; its eigenvectors are either parallel or orthogonal to $\vec{k}$. This leads to two types of modes for the isotropic vector field:

- **Longitudinal modes** correspond to deformations parallel to the wave-vector, $\hat{e}_\ell \parallel \vec{k}$, with
$$\mathbf{K}_{\alpha\beta}(\hat{e}_\ell)_\beta = (K+L)k^2(\hat{e}_\ell)_\alpha\,, \tag{3.6.41}$$
since $k_\beta(\hat{e}_\ell)_\beta = k$. The resulting dispersion relation is obtained as solution of the polynomial equation
$$-i\eta\omega - \rho\omega^2 + J = -(K+L)k^2\,. \tag{3.6.42}$$
For example, the wave-equation for $\eta = J = 0$ admits longitudinal frequencies $\omega_\ell = \pm\sqrt{(K+L)k^2/\rho}$, with wave-speed $v_\ell = \sqrt{(K+L)/\rho}$.

- **Transverse modes** have deformations perpendicular to the wave-vector, with $k_\beta(\hat{e}_t)_\beta = 0$, leading to
$$\mathbf{K}_{\alpha\beta}(\hat{e}_\ell)_\beta = Kk^2(\hat{e}_\ell)_\alpha\,. \tag{3.6.43}$$
While there is only one longitudinal direction for each $\vec{k}$, there are 2 (or $(d-1)$ in d-dimensions) transverse directions. The resulting dispersion relation satisfies
$$-i\eta\omega - \rho\omega^2 + J = -Kk^2\,. \tag{3.6.44}$$
Clearly the transverse modes of the wave-equation (with $\eta = J = 0$) have a different wave-speed of $v_t = \sqrt{K/\rho}$.

An isotropic elastic material (such as glass) admits both types of modes. A gas, however, does not respond to shear deformations and cannot support longitudinal modes. Sound waves are longitudinal pressure waves. The electromagnetic field in free space satisfies $\nabla \cdot \vec{E} = 0$ and thus cannot have a component parallel to the wave-vector. The two polarizations of electromagnetic wave are transverse to its travel direction.