3.6 Vector fields

3.6.1 Continuity equation

We arrived at the form of the Laplacian operator in circular and spherical coordinates by considering variations of $U = \int dV (\nabla h)^2/2$. It is useful to gain a different perspective by examining the problem of diffusion of a density field $n(\vec{r}, t)$. We observed in Sec. (3.3.4) that the changes in density can be attributed to a current that moves the diffusing material around. While the density is a scalar field, the diffusive current is a vector field

$$\vec{J} = -D\vec{\nabla} n \,, \tag{3.6.1}$$

which smoothens density by moving diffusers from high to low concentrations. As already noted in Eq. (3.1.19), in one dimension

$$\frac{d}{dt}\int_a^b dx n(x,t) = D\int_a^b dx \frac{\partial^2 n}{\partial x^2} = D\left[\left.\frac{\partial n}{\partial x}\right|_b - \left.\frac{\partial u}{\partial x}\right|_a\right] = J(a) - J(b)\,, \tag{3.6.2}$$

the change in density for each interval is related to the difference between incoming and outgoing currents. Given the conservation of the net diffusive substance, we should be able to construct the generalization of the above equation to higher dimensions:

- Consider a region of d-dimensional space of volume V, bounded by a surface of area S. We expect the change in the amount of diffusive material within V to be related to the net flux of material taken in or out through the surface by the current $\vec{J}$. This can be written as
$$\frac{d}{dt}\int_{\text{volume } V} dV\ n = -\int_{\text{surface } S} \vec{J}\cdot d\vec{S}\,, \tag{3.6.3}$$
where $d\vec{S}$ indicates the flux through a small element of surface of area dS; the direction of $d\vec{S}$ is taken to point outward, normal to the area element.

- For an infinitesimal volume element in the form of a hyper-cube, the above equality leads to
$$\dot{n}(\vec{r}, t)\ dV = -\sum_{i=1}^{d} dS_i \left(\frac{\partial J_i}{\partial x_i}\ dx_i\right)\,. \tag{3.6.4}$$
The contribution along each direction i comes from the difference in fluxes $\delta J_i = \frac{\partial J_i}{\partial x_i} dx_i$, passing through the orthogonal surfaces of area dS_i. For the hypercube, $dV = S_1\ dx_1 = S_2\ dx_2 = \cdots$, resulting in
$$\dot{n}(\vec{r}, t) = -\sum_{i=1}^{d} \frac{\partial J_i}{\partial x_i}\,. \tag{3.6.5}$$

- For any vector field field $\vec{v}(\vec{r})$, a scalar *divergence* is defined by (using the summation convention)
$$\text{div}\ \vec{v} = \partial_i v_i = \vec{\nabla}\cdot\vec{v}\,. \tag{3.6.6}$$

- The conservation of (in this case diffusive) material can then be expressed by the *continuity equation*,

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \vec{J} \,, \qquad \frac{\partial n}{\partial t} = D\nabla^2 n \,, \qquad \text{for} \quad \vec{J} = -D\vec{\nabla} n \,. \tag{3.6.7}$$

- While the above formulae were motivated by conservation of matter during diffusion, Eq. (3.6.1) embodies the highly important and useful *divergence theorem*

$$\int_{\text{volume } V} dV \ \text{div}\vec{v} = -\int_{\text{surface } S} \vec{v} \cdot d\vec{S} \,, \tag{3.6.8}$$

namely that the flux of a vector field $\vec{v}$ through any closed surface is equal to the integral of the divergence of $\vec{v}$ though the enclosed volume.

3.6.2 General change of coordinates

We have seen that is useful to work in a coordinate system appropriate to the properties and symmetries of the system under consideration, using polar coordinates for analyzing a circular drum, or spherical coordinates in analyzing diffusion within a sphere. For arriving at the form of the equation of motion in these coordinate systems we made use of arguments that we generalize below.

Let us start by considering the simple case of *polar coordinates*, (r, ϕ), in the 2D plane $\mathbb{R}^2$ are defined from Cartesian coordinates, (x, y) with $-\infty \leq x, y \leq \infty$, as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \begin{cases} \arccos\left(\frac{x}{r}\right) & \text{if } y \geq 0 \text{ and } r \neq 0 \\ -\arccos\left(\frac{x}{r}\right) & \text{if } y < 0 \\ \text{undefined} & \text{if } r = 0 \end{cases} \,, \tag{3.6.9}$$

and conversely

$$x = r\mathcal{O}s(\phi), \qquad y = r\sin(\phi) \,. \tag{3.6.10}$$

The same physical problem can be described in Cartesian coordinates or polar coordinates as they both span the same space (the 2D plane). If the complex number $z = x + iy$ is constructed from the Cartesian coordinates, then $z = r[\mathcal{O}s(\phi) + i\sin(\phi)] = re^{i\phi}$ and $r = |z|$ and $\phi = \arg(z)$ (defined as the principal branch).

The above equations are an example of a coordinate transformation, or change of variables. From Eq. (3.6.11) infinitesimal changes in the two sets of coordinates are related by

$$dx = dr\mathcal{O}s(\phi) - d\phi \ r\sin(\phi) \quad \text{and} \quad dy = dr\sin(\phi) + d\phi \ r\mathcal{O}s(\phi) \,. \tag{3.6.11}$$

The relation between infinitesimal changes in two coordinate systems can thus be expressed as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{J}(r, \phi) \begin{bmatrix} dr \\ d\phi \end{bmatrix} , \tag{3.6.12}$$

in terms of the so-called *Jacobian* (or Jacobi matrix)

$$\mathbf{J}(r,\phi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \mathcal{O}s\phi & -r\sin\phi \\ \sin\phi & r\mathcal{O}s\phi \end{bmatrix}. \tag{3.6.13}$$

The inverse Jacobian

$$\mathbf{J}^{-1}(x,y) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}, \tag{3.6.14}$$

provides the transformation in the reverse direction.

There are many other useful coordinate systems. Some such as spherical or cylindrical coordinates are common enough that they bear remembering, but others are specific to a particular problem. More generally, given a set of coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, a set of functions $y_i(x_1, \ldots, x_n)$ for $i = 1, \ldots n$ that map $\mathbb{R}^n \to \mathbb{R}^n$, is a coordinate transformation and changes in the new coordinates y_i are related to changes in the original coordinates through the Jacobian

$$\mathbf{J}(y_1, \ldots y_n) = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}. \tag{3.6.15}$$

The Jacobian is highly useful in computing derivatives and gradient operators in the new coordinate system:

- The change of variables transforms a function $f(\vec{x})$ in the original coordinates to a function $f(\vec{h})$ in the new set of coordinates. By application of the chain rule, the corresponding two vectors of derivatives are related by

$$\frac{\partial f}{\partial y_\alpha} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_\alpha}, \Rightarrow \quad \frac{\partial f}{\partial \vec{y}} = \mathbf{J}(\vec{y}) \cdot \frac{\partial f}{\partial \vec{x}}. \tag{3.6.16}$$

- The above can be extended to multiple functions that can be collected together as a vector. For a m-component vector function $\mathbf{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$ of an n-component coordinate vector $\vec{x}$, the matrix of derivatives with respect to a new coordinate vector $\vec{y}$ is

$$\frac{\partial \mathbf{f}}{\partial \vec{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_n} \end{bmatrix} = \mathbf{J}(\vec{y}) \cdot \frac{\partial \mathbf{f}}{\partial \vec{x}}, \tag{3.6.17}$$

  where $\frac{\partial \mathbf{f}}{\partial \vec{y}}$ and $\frac{\partial \mathbf{f}}{\partial \vec{x}}$ are $m \times n$ matrices.

- If only one of the new set of coordinates is changed, say y_1 by dy_1, from Eq. (3.6.15), the corresponding change in the original Cartesian coordinates is

$$d\vec{r}_1 = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} \\ \vdots \\ \frac{\partial x_n}{\partial y_1} \end{bmatrix} dy_1 \equiv (h_1\ dy_1)\hat{e}_1. \tag{3.6.18}$$

Through the above equation we have defined two important quantities: a unit vector $\hat{e}_1$ pointing along the direction of the change caused by dy_1; and a metric factor h_1 quantifying (as $h_1\ dy_1$) the magnitude of the change. The complete sets of $\{\hat{e}_\alpha\}$ and $\{h_\alpha\}$ quantify variations along all directions.

- The absolute value of the Jacobian determinant represents the change in volume element when making a change of variables while evaluating a integral of a function over a region. To accommodate for the change of coordinates $|\det \mathbf{J}|$ is included as a multiplicative factor within the integral. This is because the n-dimensional dV element is in general a parallelepiped in a new coordinate system made up of the infinitesimal vectors $\{(h_\alpha \hat{e}_\alpha) dy_\alpha\}$. The volume of this infinitesimal parallelepiped is obtained as the determinant of its edge vectors. For the case of Cartesian to polar coordinates, the determinant of Eq. (3.6.13) gives

$$dx\, dy = |\det \mathbf{J}| dr\, d\phi = r(\mathcal{O}s^2\phi + \sin^2\psi) dr\, d\phi = r\, dr\, d\phi\,, \tag{3.6.19}$$

a result we used previously in Eq. (3.5.1).

- The set of metric factors $\{h_\alpha\}$ and corresponding unit vectors $\{\hat{e}_\alpha\}$ can be used to construct the gradient operation in the new coordinates as

$$\vec{\nabla} f = \sum_\alpha \frac{1}{h_\alpha} \frac{\partial f}{\partial y_\alpha} \hat{e}_\alpha\,. \tag{3.6.20}$$

Clearly, along with Eq. (3.6.18), this leads to the expected $df = \frac{\partial f}{\partial y_\alpha} dy_\alpha$. For polar coordinates, Eq. (3.6.13) indicates that

$$d\vec{r} = dr\ \hat{r} + d\phi\ r\hat{\phi}\,, \quad \text{and} \quad \vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi}\,. \tag{3.6.21}$$

- To construct the divergence of a vector field $\vec{v}$, we employ the infinitesimal variant of the divergence theorem, Eq. (3.6.8), as in Eq. (3.6.4)[7]

$$\text{div}\vec{v}\ dV = \sum_\alpha dy_\alpha \frac{\partial}{\partial y_\alpha}(dS_\alpha v_\alpha)\,. \tag{3.6.22}$$

The volume and surface elements in the above formula are related by

$$dV = |\det \mathbf{J}| \prod_\alpha dy_\alpha = (h_1\ dy_1) dS_1 = (h_1\ dy_1) dS_1 = \cdots\,. \tag{3.6.23}$$

Dividing by dV thus leads to

$$\text{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{1}{|\det \mathbf{J}|}\sum_\alpha \frac{\partial}{\partial y_\alpha}\left(\frac{|\det \mathbf{J}|}{h_\alpha} v_\alpha\right)\,. \tag{3.6.24}$$

[7]The explicit inclusion of the summation symbol $\sum_\alpha$ indicates that the summation convention is no longer used.

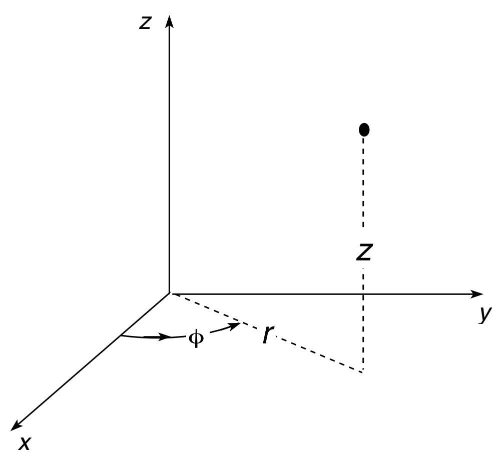
- Finally, the Laplacian of a scalar field f in general (curvilinear) coordinates is obtained as the divergence of the gradient of f, and given by

$$\nabla^2 f = \frac{1}{|\det \mathbf{J}|} \sum_\alpha \frac{\partial}{\partial y_\alpha} \left(\frac{|\det \mathbf{J}|}{h_\alpha^2} \frac{\partial f}{\partial y_\alpha} \right) . \tag{3.6.25}$$

In the next sections we shall use these general formulae to re-derive the explicit forms of these operations in two commonly encountered coordinate systems.

3.6.3 Cylindrical and spherical coordinates

Cylindrical coordinates simply extend the 2D polar coordinates, Eq. (3.6.11), by adding a third coordinate, z, pointing out of the plane.



From the Jacobian

$$\mathbf{J}(r, \phi, z) = \begin{bmatrix} \mathcal{O}s\phi & -r\sin\phi & 0 \\ \sin\phi & r\mathcal{O}s\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{3.6.26}$$

we obtain the metric factors

$$h_r = 1 , \qquad h_\phi = r, \qquad h_z = 1 , \quad \text{and} \quad |\det \mathbf{J}| = r . \tag{3.6.27}$$

For a scalar field $\Phi(r, \phi, z)$, we then obtain

$$\nabla \Phi = \left(\frac{\partial \Phi}{\partial r}, \frac{1}{r}\frac{\partial \Phi}{\partial \phi}, \frac{\partial \Phi}{\partial z} \right) , \tag{3.6.28}$$

and

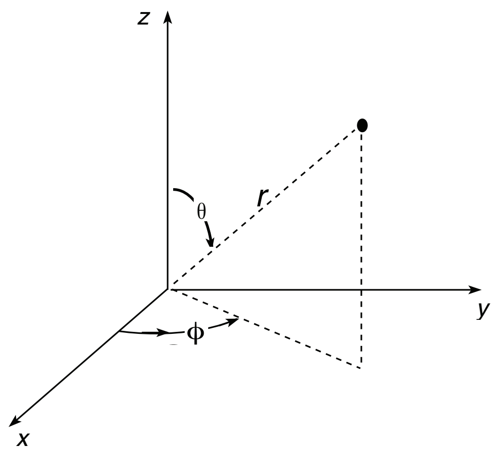
$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} . \tag{3.6.29}$$

A vector field can also be presented in these coordinates with components (v_r, v_ϕ, v_z) that are functions of (r, ϕ, z). The divergence of the vector is then obtained as

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}\,. \tag{3.6.30}$$

Spherical coordinates provide another representation of three dimensional space, replacing the axiz z with the angle θ to the z axis, such that $z = r\mathcal{O}s(\theta)$. With r now indicating the distance to the origin, its projection onto the 2D plane has length $r\sin(\theta)$, such that

$$x = r\sin(\theta)\mathcal{O}s(\phi) \quad y = r\sin(\theta)\sin(\phi) \quad z = r\mathcal{O}s(\theta)\,, \quad \text{with} \quad 0 \le \theta \le \pi \quad \text{and} 0 \le \phi < 2\pi\,. \tag{3.6.31}$$



The Jacobian associated with this transformation is

$$\mathbf{J}(r,\theta,\phi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\mathcal{O}s\phi & r\mathcal{O}s\theta\mathcal{O}s\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\mathcal{O}s\theta\sin\phi & r\sin\theta\mathcal{O}s\phi \\ \mathcal{O}s\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{3.6.32}$$

From the magnitudes of the column vectors, we find

$$h_r = 1 \qquad h_\theta = r\,, \quad h_\phi = \sin\theta, \qquad \text{while} \quad |\det \mathbf{J}| = r^2 \sin\theta\,. \tag{3.6.33}$$

For a scalar field $\Phi(r,\theta,\phi)$, we then have

$$\vec{\nabla}\Phi = \left(\frac{\partial \Phi}{\partial r}, \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right)\,, \tag{3.6.34}$$

and

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial \Phi}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}\,. \tag{3.6.35}$$

For a vector field $\vec{v} = (v_r, v_\theta, v_\phi)$,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\,. \tag{3.6.36}$$

3.6.4 Dynamics of vector Fields

We introduced scalar fields in one dimension by considering the limit of a chain of N particles connected by springs or via a rubber band. While we assumed that the particles were constrained to move along the line, with displacements quantified by $\{u_n\}$, they could have well been allowed to move in two or three dimensions, with corresponding displacements indicated by a set of vectors $\{\vec{u}_n\}$. Taking the continuum limit ($na \to x \in [0, L = Na]$) now leads to a vector field $\vec{u}(x)$.

In the above example of the chain, the coordinate x is one dimensional, while the vector field $\vec{u}(x)$ can be 2, or 3 dimensional. More generally, we can consider vector fields $\vec{u}(\mathbf{x})$ where $\vec{u} = (u_1, u_2, \cdots, u_m)$ has m components spanning a d-dimensional space of coordinates $\mathbf{x} = (x_1, x_2, \cdots, x_d)$. In the majority of cases $m = d$, as for the case of the electric and magnetic fields in three dimensions. Another example, most directly generalizing displacements of a chain to three dimensions corresponds to the distortions $\vec{u}(\vec{x})$ of a solid in $m = d = 3$ dimensions. The gradient of a scalar field is also an example of a vector field.

Generalizing the approach we have followed for particles and scalar fields, we may seek to describe the changes in the vector field as a function of time, $\vec{u}(\mathbf{x}, t)$ via a vectorial variant of PDEs as

$$\eta \frac{\partial \vec{u}}{\partial t} + \rho \frac{\partial^2 \vec{u}}{\partial t^2} + \cdots = \vec{\mathcal{F}}(\mathbf{x}) \,, \tag{3.6.37}$$

where the vector force density depends on the vector field $\vec{u}(\mathbf{x})$ and its derivatives (in a gradient expansion) around the point $\mathbf{x}$.

Let us focus on the Taylor expansion of the force $\vec{\mathcal{F}}(\mathbf{x})$ for small distortions $\vec{u}(\mathbf{x})$, say in the relevant case of a solid or gel in three dimensions. We can followi the procedure described in Sec. 3.4.1, and employing the constrains of *locality* and *uniformity* construct a *gradient expansion* for the vectorial force field. The mathematical consequence of *isotropy*, however, is more interesting: In dealing with scalar quantities the contraction of two gradient operators lead to the term $\nabla^2 h$, for vectorial quantities, one gradient can be contracted with the vector field, resulting in a new contribution to the gradient expansion. In index notation, we thus find

$$\vec{\mathcal{F}}_\alpha(\mathbf{x}) = -J u_\alpha + K \partial_\beta \partial_\beta u_\alpha + L \partial_\alpha \partial_\beta u_\beta + \cdots \,; \tag{3.6.38}$$

in addition to the usual Laplacian– now for the vector field $\vec{u}(\vec{x})$, there is a contribution that is the gradient of the divergence of the field– $\nabla(\nabla \cdot \vec{u})$.

The dispersion relation governing the PDE of Eq. (3.6.37), with the force density in Eq.(3.6.38) is obtained by considering the (vectorial) trial solution

$$u_\alpha(\vec{x}) \propto \Re[e^{-i\omega t}] \Re[e^{i\vec{k}\cdot\vec{x}}] \hat{e}_\alpha \,, \tag{3.6.39}$$

with $\hat{e}_\alpha$ indicating the α-th component of a unit vector $\hat{e}$. Substituting the above form in the equation leads to

$$\left[-i\eta\omega - \rho\omega^2 + J\right] \hat{e}_\alpha = -\left[K k^2 \delta_{\alpha\beta} + L k_\alpha k_\beta\right] \hat{e}_\beta \equiv -\mathbf{K}_{\alpha\beta} \hat{e}_\beta \,, \tag{3.6.40}$$

with a matrix relation emerging from the mixed derivative $\partial_\alpha \partial_\beta$.

While the wave-vector $\vec{k}$ does introduce a particular direction, the matrix $\mathbf{K}$ still reflects our assumption of isotropy; its eigenvectors are either parallel or orthogonal to $\vec{k}$. This leads to two types of modes for the isotropic vector field:

- **Longitudinal modes** correspond to deformations parallel to the wave-vector, $\hat{e}_\ell \parallel \vec{k}$, with
$$\mathbf{K}_{\alpha\beta}(\hat{e}_\ell)_\beta = (K+L)k^2(\hat{e}_\ell)_\alpha\,, \tag{3.6.41}$$
since $k_\beta(\hat{e}_\ell)_\beta = k$. The resulting dispersion relation is obtained as solution of the polynomial equation
$$-i\eta\omega - \rho\omega^2 + J = -(K+L)k^2\,. \tag{3.6.42}$$
For example, the wave-equation for $\eta = J = 0$ admits longitudinal frequencies $\omega_\ell = \pm\sqrt{(K+L)k^2/\rho}$, with wave-speed $v_\ell = \sqrt{(K+L)/\rho}$.

- **Transverse modes** have deformations perpendicular to the wave-vector, with $k_\beta(\hat{e}_t)_\beta = 0$, leading to
$$\mathbf{K}_{\alpha\beta}(\hat{e}_\ell)_\beta = Kk^2(\hat{e}_\ell)_\alpha\,. \tag{3.6.43}$$
While there is only one longitudinal direction for each $\vec{k}$, there are 2 (or $(d-1)$ in d-dimensions) transverse directions. The resulting dispersion relation satisfies
$$-i\eta\omega - \rho\omega^2 + J = -Kk^2\,. \tag{3.6.44}$$
Clearly the transverse modes of the wave-equation (with $\eta = J = 0$) have a different wave-speed of $v_t = \sqrt{K/\rho}$.

An isotropic elastic material (such as glass) admits both types of modes. A gas, however, does not respond to shear deformations and cannot support longitudinal modes. Sound waves are longitudinal pressure waves. The electromagnetic field in free space satisfies $\nabla \cdot \vec{E} = 0$ and thus cannot have a component parallel to the wave-vector. The two polarizations of electromagnetic wave are transverse to its travel direction.

Recap

- The *divergence theorem* relates the flux of a vector field $\vec{v}$ through any closed surface to the integral of the divergence of $\vec{v}$ though the enclosed volume:
$$\int_{\text{volume } V} dV\ \text{div}\vec{v} = -\int_{\text{surface } S} \vec{v}\cdot d\vec{S}\,. \tag{3.6.45}$$

- For a general coordinate system, the expressions for divergence of a vector field and the Laplacian of a scalar field are given by
$$\text{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{1}{|\det\mathbf{J}|}\sum_\alpha \frac{\partial}{\partial y_\alpha}\left(\frac{|\det\mathbf{J}|}{h_\alpha}v_\alpha\right)\,,$$
$$\nabla^2 f = \frac{1}{|\det\mathbf{J}|}\sum_\alpha \frac{\partial}{\partial y_\alpha}\left(\frac{|\det\mathbf{J}|}{h_\alpha^2}\frac{\partial f}{\partial y_\alpha}\right)\,. \tag{3.6.46}$$

- In spherical coordinates, the corresponding expressions are

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\,, \\
\vec{\nabla}\cdot\vec{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,.
\end{aligned}
\tag{3.6.47}
$$

- The isotropic vector field in d-dimensions admits one longitudinal mode parallel to the wave-vector, and $(d-1)$ transverse modes perpendicular to it.