4.1.4 Poisson distribution

The classical example of a Poisson process is radioactive decay. Observing a piece of radioactive material over a time interval T shows that:

- The probability of one and only one event (decay) in the interval $[t, t+dt]$ is proportional to dt as $dt \to 0$,
- The probabilities of events at different intervals are independent of each other.

The probability of observing exactly M decays in the interval T is given by the Poisson distribution. It is obtained as a limit of the binomial distribution by subdividing the interval into $N = T/dt \gg 1$ segments of size dt. In each segment, an event occurs with probability $p = \alpha dt$, and there is no event with probability $q = 1 - \alpha dt$. As the probability of more than one event in dt is too small to consider, the process is equivalent to a binomial one. Using Eq. (4.1.10) the generating function for this process is obtained as

$$G(\lambda) = \left(pe^{\lambda} + q\right)^{n} = \lim_{dt\to 0}\left[1 + \alpha dt\left(e^{\lambda} - 1\right)\right]^{T/dt} = \exp\left[\alpha(e^{\lambda} - 1)T\right] \quad . \tag{4.1.13}$$

The cumulants of the distribution are obtained from the expansion

$$\ln G(\lambda) = \alpha T(e^{\lambda} - 1) = \alpha T \sum_{n=1}^{\infty} \frac{(\lambda)^n}{n!}, \quad \Longrightarrow \quad \langle M^n \rangle_c = \alpha T \quad . \tag{4.1.14}$$

All cumulants have the same value, and the moments are obtained as

$$\langle M \rangle = (\alpha T), \quad \left\langle M^2 \right\rangle = (\alpha T)^2 + (\alpha T), \quad \left\langle M^3 \right\rangle = (\alpha T)^3 + 3(\alpha T)^2 + (\alpha T). \tag{4.1.15}$$

Using a similar limiting procedure on the binomial distribution leads to

$$p_{\alpha T}(M) = e^{-\alpha T}\frac{(\alpha T)^M}{M!} \,. \tag{4.1.16}$$

Example: Assuming that stars are randomly distributed in the galaxy (clearly unjustified) with a density n, what is the probability that the nearest star is at a distance R?

Since, the probability of finding a star in a small volume dV is ndV, and they are assumed to be independent, the number of stars in a volume V is described by a Poisson process as in Eq. (4.1.16), with $\alpha = n$. The probability $p(R)$, of encountering the first star at a distance R is the product of the probabilities $p_{nV}(0)$, of finding zero stars in the volume $V = 4\pi R^3/3$ around the origin, and $p_{ndV}(1)$, of finding one star in the shell of volume $dV = 4\pi R^2 dR$ at a distance R. Both $p_{nV}(0)$ and $p_{ndV}(1)$ can be calculated from Eq. (4.1.16), and

$$\begin{aligned} p(R)dR = p_{nV}(0)\, p_{ndV}(1) =& e^{-4\pi R^3 n/3}\, e^{-4\pi R^2 ndR}\, 4\pi R^2 ndR, \\ \Longrightarrow \qquad p(R) =& 4\pi R^2 n \exp\left(-\frac{4\pi}{3}R^3 n\right) \quad . \end{aligned} \tag{4.1.17}$$