4.1 Random Variable

4.1.1 Describing random change

While motion in the realm of macroscopic bodies is largely deterministic, in the realm of microscopic motion, stochasticity is the rule. A prominent classical example is provided by motion of colloids in a viscous fluid. A large stone drops in such a fluid falls with a uniform (terminal) velocity set by the balance of the force of gravity and fluid friction. A micron-sized particle in the fluid, however, if observed with a microscope, performs a jittery motion that only on average moves in the direction of gravity. The Scottish botanist, Robert Brown, first discussed these fluctuations in 1827. To discuss and analyze such *Brownian motion* a new mathematical perspective is needed, based on the concept of probability, which is the topic of the next part of this material.

The simplest example of a random variable is a coin toss that can come up head or tails. More generally, a random variable may have a set of possible outcomes $\mathcal{S} \in \{x_1, x_2, \cdots, x_n\}$, e.g. $n = 6$ for outcomes of throwing a dice. To various outcomes of the random variable, we then assign probabilities, which must satisfy the following conditions:

- *Positivity*: $p_i \geq 0$, i.e. all probabilities must be real and non-negative.
- *Additivity*: Probabilities of independent outcomes is additive, e.g. the probability of an even number in the throw of a dice is the sum of probabilities for obtaining 2, 4, and 6.
- *Normalization*: $p(\mathcal{S}) = \infty$, i.e. the random variable must have take one of the possible set of outcomes.

In principle, there are two approaches to assigning probabilities:

- *Objective* probabilities are obtained *experimentally* from the relative frequency of the occurrence of an outcome in many tests of the random variable. If the random process is repeated N times, and the event A occurs N_A times, then
$$p(A) = \lim_{N\to\infty} \frac{N_A}{N}.$$
For example, a series of $N = 100,\ 200,\ 300$ throws of a dice may result in $N_1 = 19,\ 30,\ 48$ occurrences of 1. The ratios .19, .15, .16 provide an increasingly more reliable estimate of the probability $p_{\text{dice}}(\{1\})$.
- *Subjective* probabilities provide a *theoretical* estimate based on the uncertainties related to lack of precise knowledge of outcomes. For example, the assessment $p_{\text{dice}}(\{1\}) = 1/6$, is based on the knowledge that there are six possible outcomes to a dice throw, and that in the absence of any prior reason to believe that the dice is biased, all six are equally likely. The consequences of such subjective assignments of probability have to be checked against measurements, and they may need to be modified as more information about the outcomes becomes available.